Slopes of overconvergent modular forms

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For my Parents and my Grandparents

Abstract

In the first part of this thesis, we consider the slopes of the U_2 operator acting on spaces of 2-adic overconvergent modular forms with nontrivial weight-character of tame level 1.

We establish a sufficient criterion for these slopes to be given by a simple formula, and prove this criterion in several cases. This allows us to write down all of the slopes for certain weight-characters of small level. This criterion can be written quite simply in terms of modular functions.

These results on overconvergent modular forms also imply similar results for certain spaces of classical modular forms, which also allows us to prove results about the field over which the Fourier expansions of the normalised classical cuspidal modular eigenforms are defined.

These calculations provide evidence for an analogue of the Gouvêa-Mazur conjecture that the slopes of classical cuspidal modular eigenforms vary smoothly as the weight varies.

We also present new conjectures of the same form for $p = 3$ and $p = 5$, and present some numerical evidence for them.

The second part of this thesis considers the Hecke algebras attached to certain spaces of classical cuspidal modular forms of prime level. It proves that some of these Hecke algebras are not Gorenstein when localised at a prime ideal above 2. This shows that the methods developed by Mazur, Gross and Edixhoven for proving that localisations of a Hecke algebra are Gorenstein fail in some cases, because we have exhibited explicit localisations which are not Gorenstein.

The computations in both parts of the thesis show the power of computational methods when applied to number theory, in solving problems which can be described concretely. It also shows that computational methods can be useful in identifying patterns and generating data. These can then be investigated by more theoretical methods.

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Declaration

I declare that the results of this thesis are my own, and that this thesis is my own work in my own words, except for Chapter 5. The theorems there are my own work, but the final exposition is that of Kevin Buzzard, except for Section 5.5, which is my own work in my own words.

Lloyd Kilford

I declare that the above statement is accurate.

Kevin Buzzard

Contents

Part I

Overconvergent 2-adic modular forms

Chapter 1

Definitions and previous work

1.1 Slopes of modular forms

In this thesis, we will prove the following theorem:

Theorem 1.1 *Let τ be the nontrivial character of conductor 4, and let k be a positive integer. The slopes of the U_2 operator acting on $S_{2k+1}(\Gamma_0(4), \tau)$ are*

$$2, 4, 6, \ldots, 2k-2.$$

Let χ be the even primitive Dirichlet character of conductor 8. The slopes of the U_2 operator acting on $S_{2k+1}(\Gamma_0(8), \chi \cdot \tau)$ are

$$1, 2, 3, \ldots, 2k-1.$$

The slopes of the U_2 operator acting on $S_{4k+2}(\Gamma_0(8), \chi)$ are

$$1, 2, 3, \ldots, 4k.$$

We will also prove the following corollary:

Corollary 1.2 *Let k be a non-negative integer, and let S be one of the three spaces of modular forms: $S_{2k+1}(\Gamma_0(4), \tau)$, $S_{2k+1}(\Gamma_0(8), \chi\tau)$ or $S_{4k+2}(\Gamma_0(8), \chi)$.*
Let $f \in S$ be a normalised eigenform. Then the coefficients of the Fourier expansion of f are elements of $\mathbf{Q}_2$.

1.2 Definitions

We fix a rational prime p, and fix the notation for certain rings and fields that we will be using.

We write the p-adic integers as $\mathbf{Z}_p$, the field of p-adic numbers as $\mathbf{Q}_p$, and the completion of the algebraic closure of $\mathbf{Q}_p$ as $\mathbf{C}_p$. We write the integers of $\mathbf{C}_p$ as $\mathbf{O}_p$. We define the natural numbers $\mathbf{N}$ to be $\{1, 2, \dots\}$; in other words $0 \notin \mathbf{N}$.

We will now define integral weight-characters.

Definition 1.3 *Fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$.*
Let $\chi : (\mathbf{Z}/p^n\mathbf{Z})^* \to \mathbf{C}^*$ be a Dirichlet character. We note that the image of χ lies in $\overline{\mathbf{Q}}^*$, and therefore that χ can be considered as a map to $\mathbf{C}_p^*$. Let k be an integer, such that $\chi(-1) = (-1)^k$. A continuous group homomorphism of the form

$$\begin{array}{rcl} \mathbf{k} : \mathbf{Z}_p^* & \to & \mathbf{C}_p^* \\ x & \mapsto & x^k \cdot \chi(x) \end{array}$$

is called an integral weight-character of tame level 1. *We write it as $\mathbf{k} = (k, \chi)$.*
We define $\mathbf{1}$ to be the trivial Dirichlet character, which has conductor 1. We will call the trivial weight-character $\mathbf{0} := (0, \mathbf{1})$.

In Coleman-Mazur [12], weight-characters of this form are called *accessible weights with coordinates* (χ, k). We note that the definition of weight-space given in [12], section 1.4, is more general, and is defined for tame level greater than 1, but in this thesis we will only consider integral weight-characters of tame level 1.

We now introduce the notion of a modular form over an arbitrary ring, using the definition of Katz [23].

If $s : E \to S$ is an elliptic curve E over an affine scheme S, then we let $\underline{\omega}_{E/S}$ be the invertible sheaf $s_*\left(\Omega^1_{E/S}\right)$ on S.

Definition 1.4 (Katz, [23], Section 1.2, Gouvêa, [19], Section I.1) *Let k be a positive integer.*
A meromorphic (at infinity) modular form of weight k *is a rule which assigns to any elliptic curve E over any scheme S a section $f(E/S)$ of $\left(\underline{\omega}_{E/S}\right)^{\otimes k}$ in a way which only depends on the S-isomorphism class of the elliptic curve E/S, and which commutes with arbitrary base change $g : S' \to S$.*

We define meromorphic modular forms over a ring in the following way:

Definition 1.5 (Katz, Section 1.2) *Let R_0 be a ring.*
A meromorphic modular form of weight k defined over R_0 *is a rule which assigns to each elliptic curve E over any scheme S lying over R_0 a section $f(E/S)$ of $\left(\underline{\omega}_{E/S}\right)^{\otimes k}$ in a way which only depends on the isomorphism class of E/S, and which commutes with arbitrary base change by an R_0-morphism $g : S' \to S$.*

We will now define the q-expansions of a meromorphic modular form defined over R_0.

Definition 1.6 (Katz, Section 1.2) *Let k be a positive integer and let R_0 be a ring. Let* $\mathrm{Tate}(q)$ *be the Tate curve, and let ω_{can} be its canonical differential. Let f be a meromorphic modular form of weight k defined over R_0.*
*We define the q-*expansion of f at ∞ *to be the finite-tailed Laurent series*

$$f(\mathrm{Tate}(q), \omega_{can}) \in \mathbf{Z}((q)) \otimes_{\mathbf{Z}} R_0.$$

We now define holomorphic modular forms and cusp forms of weight k over a ring R.

Definition 1.7 (Katz, Section 1.2) *Let f be a meromorphic modular form of weight k defined over a ring R_0. We say that f is a* holomorphic modular form, *or simply a* modular form, *if its q-expansion lies in $\mathbf{Z}[[q]] \otimes_{\mathbf{Z}} R_0$.*
If f is a holomorphic modular form, and the q-expansion of f has zero constant term, then we say that f is a cusp form.

We remark here that Definitions 1.4–1.7 are only valid for level 1 modular forms. If the modular forms have more level structure, then this level structure becomes a part of the definition.
If M is an integer and S is a scheme, define $\mu_{M/S}$ to be the finite flat group scheme over S given by the kernel of multiplication by M in the multiplicative group over S.

Definition 1.8 (Katz, Section 1.3) *We define a* modular form of weight k on $\Gamma_1(p^m)$ *to be a rule f which assigns to each pair $(E/S, \alpha)$ consisting of an elliptic curve E over a scheme S on which p is invertible and an injection $\alpha : \mu_{p^m} \hookrightarrow \mathcal{E}$ over the base scheme S a section $f(E/S, \alpha)$ of $\left(\omega_{E/S}\right)^{\otimes k}$ over S which only depends on the isomorphism class of $(E/S, \alpha)$ and which commutes with arbitrary change of base.*
Let R_0 be a ring in which p is invertible. We define a modular form of weight k on $\Gamma_1(p^m)$ defined over R_0 *to be a rule f which assigns to each pair $(E/S, \alpha)$ consisting of an elliptic curve E over a scheme S lying over R_0 and an injection $\alpha : \mu_{p^m} \hookrightarrow \mathcal{E}$ over the base scheme S a section $f(E/S, \alpha)$ of $\left(\omega_{E/S}\right)^{\otimes k}$ over S which only depends on the isomorphism class of $(E/S, \alpha)$ and which commutes with arbitrary change of base.*

We now define modular forms with weight-character $\mathbf{k}$.

Definition 1.9 (Gouvêa [19], Section I.3.4, Katz [23], Section 1.3) *Let ζ_{p^m} be a primitive $(p^m)^{\mathrm{th}}$ root of unity. Let f be a modular form of weight k on $\Gamma_1(p^m)$ defined over an extension of $\mathbf{Z}[1/p, \zeta_{p^m}]$.*
Let E be an elliptic curve over a scheme S and let $\mathbf{k} = (k, \chi)$ be a weight-character such that the conductor of χ is p^m.

Let R be a ring which is an extension of $\mathbf{Z}[1/p, \zeta_{p^m}]$. If $S = \mathrm{Spec}(R)$ and $\underline{\omega}_{E/R}$ is a free R-module with basis element ω, then we define an element $f(E/R, \omega, \alpha)$ by

$$f(E/\mathrm{Spec}(R), \alpha) = f(E/R, \omega, \alpha) \cdot \omega^{\otimes k}.$$

We define an action of $(\mathbf{Z}/(p^m\mathbf{Z}))^\times$ on the space of modular forms of weight k on $\Gamma_1(p^m)$ by

$$f|\langle x\rangle(E, \omega, \alpha) = f(E, \omega, x \cdot \alpha),$$

where $x \in (\mathbf{Z}/(p^m\mathbf{Z}))^\times$ acts on α by sending an element $\zeta \in \mu_{p^m}$ to ζ^x.
We say that f has weight-character $\mathbf{k} := (k, \chi)$ *if, for all $x \in (\mathbf{Z}/(p^m\mathbf{Z}))^\times$, we have*

$$f|\langle x\rangle(E, \omega, \alpha) = \chi(x) \cdot f(E, \omega, \alpha).$$

There are linear operators defined on spaces of modular forms, called the *Hecke Operators*. Let l be a prime not equal to p. We will recall the action of the operators T_l, U_p and V_p on q-expansions here.

Theorem 1.10 (Katz [23], Section 1.11, Gouvêa [19], Section II.1.1) *Let $\mathbf{k} = (k, \chi)$ be a weight-character, with the conductor of χ being p^m.*
Let f be a modular form of weight-character $\mathbf{k}$ on $\Gamma_1(p^m)$, defined over a finite extension of $\mathbf{Q}_p[\zeta_{p^m}]$, with q-expansion at the cusp ∞ equal to

$$f(q) = \sum_{i=0}^{\infty} a_i q^i.$$

The action of the operator T_l on the q-expansion of f at ∞ is given by

$$(T_l f)(q) = \sum_{i=0}^{\infty} \left(a_{l\cdot i} + \chi(l) l^{k-1} a_{i/l}\right) q^i,$$

where $a_{i/l}$ is defined to be 0 if l does not divide i.
The action of the operator U_p on the q-expansion of f at ∞ is given by

$$(U_p f)(q) = \sum_{i=0}^{\infty} a_{p\cdot i} q^i.$$

The action of the operator V_p on the q-expansion of f at ∞ is given by

$$(V_p f)(q) = \sum_{i=0}^{\infty} a_i q^{ip}.$$

If f is a modular form, then so are $T_l(f)$, $U_p(f)$ and $V_p(f)$. If f is a cusp form, then so are $T_l(f)$,

$U_p(f)$ and $V_p(f)$.
If f is a modular form on $\Gamma_1(p^m)$, then $U_p(f)$ is a modular form on $\Gamma_1(p^m)$, and $V_p(f)$ is a modular form on $\Gamma_1(p^{m+1})$.

We say that a modular form is *normalised* if the coefficient of q^1 in the q-expansion of f at ∞ is equal to 1.
We say that a normalised modular form f over R is an *eigenform for the Hecke operator U_p* if we have that

$$(U_p f)(q) = \lambda_p f(q)$$

for some $\lambda_p \in R$.
Let $l \neq p$ be a prime. We say that a normalised modular form f defined over R is an *eigenform for the Hecke operator T_l* if

$$(T_l f)(q) = \lambda_l f(q),$$

for some element λ_l of R.
If f is a normalised modular form which is an eigenform for all of the Hecke operators U_p and T_l simultaneously, then we say simply that it is an *eigenform*.
We define the *slope* of a normalised modular eigenform.

Definition 1.11 (Gouvêa-Mazur [18], page 795) *Let $v_p : \mathbf{Q}_p \to \mathbf{Q} \cup \{\infty\}$ be the valuation map, normalised so that $v_p(p) = 1$.*
This extends uniquely to any extension K of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$.
The p-slope *of a normalised modular eigenform f defined over K is the p-valuation of the* p^{th} *coefficient of the q-expansion of f at ∞.*
If it is clear from context which p the valuation is taken with respect to, then we will call these simply the valuation *and* the slope*, and write v_p as simply v.*

We recall the definition of eigensubspaces of the space of holomorphic modular forms of a given weight and level from [18], page 795. We note that it is possible to define the notion of a modular form of level $\Gamma_1(Np^n)$, in a similar way to the definitions given above. We refer to [19], Chapter 1, for details.

Definition 1.12 *Let L be an extension field of $\mathbf{Q}_p$ contained in $\mathbf{C}_p$ which contains the* p^{th} *roots of unity, let N be a positive integer not divisible by p, and let $S(L; N, k)$ be the finite-dimensional L-vector space of cusp forms of level $\Gamma_1(Np)$ and weight-character $(k, \mathbf{1})$ defined over L. There is a direct sum decomposition given by the action of the U_p operator on $S(L; N, k)$:*

$$S(L; N, k) = \bigoplus_{0 \leq \alpha \in \mathbf{Q}} S(L; N, k)^{\alpha}$$

We call $S(L;N,k)^\alpha$ *the* slope α eigensubspace *of* $S(L;N,k)$*; it is defined in [18] as the image of* $P_\alpha(U_p)$ *acting upon* $S(L;N,k)$*, where* $P_\alpha(T)$ *is the factor of the characteristic polynomial of* U_p *on* $S(L;N,k)$ *which contains all eigenvalues of* U_p *whose valuation is not* α*.*

1.3 The Gouvêa-Mazur Conjecture

Gouvêa and Mazur made the following conjecture in their paper [18]:

Conjecture 1.13 (Gouvêa-Mazur [18], Conjecture 1) *Let* α *be a non-negative rational number, and let* N *be a positive integer which is not divisible by* p*. Let* k_1*,* k_2*,* n *be integers such that* $n > 0$*, and such that*

$$k_1 \equiv k_2 \mod (p-1)\cdot p^n, \text{ with } \alpha \leq n \text{ and } k_1, k_2 \geq 2\alpha+2, \text{ if } p > 2,$$

or

$$k_1 \equiv k_2 \mod 2^{n-1}, \text{ with } \alpha \leq n \text{ and } k_1, k_2 \geq 2\alpha+2, \text{ if } p = 2.$$

Then the following relation holds:

$$\dim S(\mathbf{C}_p;N,k_1)^\alpha = \dim S(\mathbf{C}_p;N,k_2)^\alpha,$$

where $S(\mathbf{C}_p;N,k_i)^\alpha$ *is the slope* α *eigensubspace of* $S(\mathbf{C}_p;N,k_i)$*.*

Gouvêa and Mazur present numerical evidence for their conjecture in [18].
We note that some authors such as Wan assume $p \geq 5$ to avoid problems with the small primes 2 and 3. Other conjectures in the original article by Gouvêa and Mazur [18] specifically exclude the cases where $p = 2$ and $p = 3$.
For a given prime p and weight-character $\mathbf{k}$, we will henceforth assume that the set of slopes of modular eigenforms of weight-character $\mathbf{k}$ is ordered, beginning with the smallest. Therefore, if we say “the first slope”, we mean the *smallest* slope.

1.4 Previous Work

There have been attempts to prove the Gouvêa-Mazur Conjecture: weaker versions are known, such as

Theorem 1.14 (Wan, [35]) *Let* $p \geq 5$ *and let* N *be an integer greater than 1 such that* $(N,p) = 1$*. If* k_1 *and* k_2 *are non-negative integers greater than or equal to* $2\alpha+2$*, then there exists* $m(\alpha,N)$*,* *which is an explicit quadratic function of α and N, such that if*

$$k_1 \equiv k_2 \mod p^{m(\alpha,N)} \cdot (p-1),$$

then

$$\dim S(\mathbf{C}_p; N, k_1)^\alpha = \dim S(\mathbf{C}_p; N, k_2)^\alpha,$$

where $S(\mathbf{C}_p; N, k_i)^\alpha$ is the slope α eigensubspace of $S(\mathbf{C}_p; N, k)$
The proof of this given in [35] shows also that $m(\alpha, N)$ depends only on N and α.

Another approach to the problem has been to try to determine the lowest slope(s) for a given prime p and weight-character $\mathbf{k}$. Emerton [16] proves that

Theorem 1.15 (Emerton, "Main Theorem" (Theorem 1.1)) *Let $p = 2$, and k be an even integer such that $k \geq 12$ and $k \neq 14$. Then we have that, for level 1 and weight k modular forms defined over $\mathbf{Q}_2$:*

1. *If $k \equiv 0, 4, 8, 12 \mod 16$, the minimal slope is 3,*
2. *If $k \equiv 2, 10 \mod 16$, the minimal slope is 4,*
3. *If $k \equiv 6 \mod 16$, the minimal slope is 5,*
4. *If $k \equiv 14 \mod 16$, the minimal slope is 6.*

He also determines what the lowest slopes are for cuspidal modular forms with nontrivial character:

Theorem 1.16 (Emerton, Proposition 5.1) *Let $m > 1$ be an integer. Let $\mathbf{k} = (k, \chi)$ be a weight-character, with χ having conductor 2^m. The minimal slope for modular forms on $\Gamma_1(2^m)$ defined over $\mathbf{Q}_2(\chi(5))$ with weight-character $\mathbf{k}$ is 2^{3-m}.*

Coleman, Stevens and Teitelbaum [13] prove a very similar formula for $p = 3$:

Theorem 1.17 (Coleman, Stevens, Teitelbaum) *Let $p = 3$, and let τ_3 be the Teichmuller character. If $k \not\equiv 10, 14 \mod 9$, then the minimal slope for a classical cuspidal eigenform of weight-character (k, τ_3^{-k}) defined over $\mathbf{Q}_3$ is*

$$2 + v_3(k - 10) + v_3(k - 14).$$

One final approach is to consider what the finite sums of the first n slopes are. This can give weaker partial results on what the slopes are, by looking at the action of the U_p operator on spaces of modular forms of a fixed weight-character.
Smithline [32], using this technique, proves that

Theorem 1.18 (Smithline) *The sum of the first n slopes of 3-adic overconvergent modular forms of weight 0 and level 1 is at least $\frac{3n^2+n}{2}$, and is equal to that if $n = \frac{3^i-1}{2}$ for some i.*

Buzzard and Calegari [8] have proved the following theorem, which gives a formula of the i^{th} slope of 2-adic overconvergent modular forms of weight 0.

Theorem 1.19 (Buzzard-Calegari) *The i^{th} slope of overconvergent 2-adic modular forms of weight 0 and level 1 is given by:*

$$1 + 2 \cdot v_2\left(\frac{(3i)!}{i!}\right).$$

This formula was conjectured to hold by careful analysis of the matrix which gave the slopes, and proved by combinatorical methods. It is thought that similar methods should be applicable to other cases.

We will mention one of Calegari's conjectures in Chapter 4.

Chapter 2

The Main Theorem on matrices

2.1 Eisenstein series

We will use noncuspidal eigenforms with nonzero constant terms in their Fourier expansions at ∞ to create weight 0 modular functions. We will then evaluate the action of the U_2 operator on these forms.

To fix notation, we recall the definition of the *extended Bernoulli numbers* from Washington [36]:

Definition 2.1 *Let χ be a primitive Dirichlet character of conductor N. The* extended Bernoulli numbers $B_{m,\chi}$ *are defined by the relation*

$$\sum_{a=1}^{N} \frac{\chi(a) \cdot t \cdot \mathbf{e}^{at}}{\mathbf{e}^{Nt} - 1} = \sum_{i=0}^{\infty} B_{i,\chi} \cdot \frac{t^i}{i!}.$$

We will now define the Eisenstein series of weight-character $\mathbf{k}$.

Definition 2.2 *Let χ be a primitive Dirichlet character of conductor $p^m \geq p^2$, let k be a positive integer and let $\mathbf{k} = (k, \chi)$ be a weight-character as defined in chapter 1.*

We define the coefficient λ, the modular forms $E^*_{\mathbf{k}}$ and $V^*_{\mathbf{k}}$, and the quotient of modular forms x:

$$\begin{aligned}
\lambda &:= \frac{-B_{k,\chi}}{2k}, \\
E^*_{\mathbf{k}} &:= \lambda + \sum_{n=1}^{\infty} \left(\sum_{\substack{0<d|n \\ (d,p)=1}} \mathbf{k}(d) \cdot d^{-1} \right) \cdot q^n, \\
V^*_{\mathbf{k}} &:= V(E^*_{\mathbf{k}}) = \lambda + \sum_{n=1}^{\infty} \left(\sum_{\substack{0<d|n \\ (d,p)=1}} \mathbf{k}(d) \cdot d^{-1} \right) \cdot q^{p \cdot n}, \\
x &:= \lambda \cdot (E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1).
\end{aligned}$$

Let ζ_s be a primitive s^{th} root of unity.

We choose and fix an element μ of $\mathbf{Q}_p(\zeta_{p^{m-1}})$, such that

$$\mu^2 = \frac{1}{\lambda}.$$

If $p = 2$, then $E^*_{\mathbf{k}}$ is a modular form of weight-character $\mathbf{k}$ which is defined over $\mathbf{Q}_2(\zeta_{2^m})$.

If $p > 2$, then $E^*_{\mathbf{k}}$ is a modular form of weight-character $\mathbf{k}$ which is defined over $\mathbf{Q}_p(\zeta_{p^{m-1}})$.

We will now prove a result about the valuation of λ, when $p = 2$.

Lemma 2.3 *Let $p = 2$. Let k be a positive integer, let $m \geq 2$ be an integer and let χ be a primitive Dirichlet character of conductor 2^m. Let $\mathbf{k} = (k, \chi)$ be a weight-character. Then*

$$v_2(\lambda) = -2^{3-m}.$$

We will refer to Washington [36]. In chapter 7, exercise 7.6.d, Washington shows that, if χ has conductor 4, then the valuation of λ is 2.

Therefore, we will be considering $m \geq 3$. From chapter 7, exercise 7.7 of [36], we see that $\lambda = 1/(1 - \chi(5)^{-1}) + t$, where t is an integer.

Now $\chi(5)$ is a primitive $(2^{m-2})^{\text{th}}$ root of unity, so it is an element of the $(2^{m-2})^{\text{th}}$ cyclotomic field. We see that $\chi(5)^{-1}$ is also a $(2^{m-2})^{\text{th}}$ root of unity. From Washington, Proposition 2.14, we see that $(1 - \chi(5))$ is a prime ideal which is totally ramified in $\mathbf{Q}_2(\chi(5))$, and therefore that the valuation of λ is -2^{3-m}. ∎

We record a simple observation on these Eisenstein series here:

Lemma 2.4 *Let χ be a primitive Dirichlet character of conductor p^m, let k be a positive integer, and let $\mathbf{k} = (k, \chi)$ be a weight-character.*

The Eisenstein series $E^*_{\mathbf{k}}$ is an eigenform for the U_p operator, with eigenvalue 1.

This can be seen to be true, because the n^{th} and $(pn)^{\text{th}}$ coefficients of $E^*_{\mathbf{k}}$ are the same; we see that only the divisors of n and pn which are not divisible by p contribute to the character sum in the definition of $E^*_{\mathbf{k}}$, and these are the same for n and for pn. ∎

We also see that

$$U_p(V^*_{\mathbf{k}}) = U_p(V(E^*_{\mathbf{k}})) = E^*_{\mathbf{k}}.$$

This follows from the definitions of the U_p and V_p operators; the coefficients of the q-expansion of $E^*_{\mathbf{k}}$ are left unchanged by the operators when applied in this order:

$$U_p\left(\lambda+\sum_{n=1}^{\infty}\left(\sum_{\substack{0<d|n\\(d,p)=1}}\mathbf{k}(d)\cdot d^{-1}\right)\cdot q^{p\cdot n}\right)=\lambda+\sum_{n=1}^{\infty}\left(\sum_{\substack{0<d|n\\(d,p)=1}}\mathbf{k}(d)\cdot d^{-1}\right)\cdot q^{n}=E^*_{\mathbf{k}}.$$

2.2 Overconvergent 2-adic modular forms

In this section, and for the rest of the chapter, we will consider only overconvergent 2-adic modular forms, so we will work with $p = 2$ from now on. We fix an integer $m \geq 2$.

Definition 2.5 (Coleman-Mazur [12], Section 2.1) *Let M be an integer and let S be a scheme. Define $\mu_{M/S}$ to be the finite flat group scheme over S given by the kernel of multiplication by M in the multiplicative group (over S).*
Let $Y_1(2^m)$ be the modular curve classifying $(\mathcal{E},\alpha)$, where $\mathcal{E}$ is an elliptic curve over a $\mathbf{Z}[1/2]$-scheme S, and $\alpha : \mu_{2^m} \hookrightarrow \mathcal{E}$ is an injection over the base scheme S.
We let $X_1(2^m)$ be the compactification of $Y_1(2^m)$.

There is a universal family of elliptic curves over $Y_1(2^m)$:

$$(u : \mathbf{E}_1(2^m) \to (Y_1(2^m), \alpha)).$$

We will now define line bundles on these modular curves.

Definition 2.6 (Coleman-Mazur [12], Section 2.1) *Define ω to be the invertible line bundle on the modular curve $Y_1(2^m)_{/\mathbf{Q}_2}$ given by*

$$\omega := u_*\Omega^1_{\mathbf{E}_1(2^m)/Y_1(2^m)}.$$

We now define the Hasse invariant of an elliptic curve.

Definition 2.7 (Katz [23], Section 2.1) *Let E be an elliptic curve over an $\mathbf{F}_2$-algebra R.*
If ω is a basis element of $H^0(E,\Omega^1_{E/R})$, then we write the dual basis as $\eta \in H^1(E,\mathcal{O}_E)$. The absolute Frobenius map, $F_{\text{abs}} : \mathcal{O}_E \to \mathcal{O}_E$, given by $x \mapsto x^p$, induces an endomorphism of $H^1(E,\mathcal{O}_E)$, which we call F^*_{abs}.
We define the Hasse Invariant *$A(E)$ by $F^*_{\text{abs}}(\eta) = A(E)\cdot\eta$.*

We recall that Katz [23], section 2.1 shows that the q-expansion of $A(E)$ over $\mathbf{F}_2$ is 1.

We consider the Eisenstein series of weight 4 and tame level 1 defined over $\mathbf{Z}$, with q-expansion

$$E_4(q) := 1 + 240\sum_{n=1}^{\infty}\left(\sum_{0<d|n} d^3\right)\cdot q^n.$$

We see that E_4 is a lifting of $A(E)^4$ to characteristic 0, as the reduction of E_4 to characteristic 2 has the same q-expansion as $A(E)^4$, and therefore $E_4 \mod 2$ and $A(E)^4$ are both modular forms of level 1 and weight 4 defined over $\mathbf{F}_2$, with the same q-expansion.

We will now define the valuation of $E_4(E)$, where E is an elliptic curve together with an injection from μ_{2^m} into E.

We fix $t \in Y_1(2^m)(\overline{\mathbf{Q}}_2)$. Then t corresponds to an elliptic curve E defined over a finite extension K of $\mathbf{Q}_2$ (together with an injection from μ_{2^m} into E). Therefore, we see that

$$E_4(E) \in H^0(E, \Omega^1_{E/K})^{\otimes 4} \cong K.$$

If E does not have good reduction, then we define $v(E_4(E))$ to be 0.

If E does have good reduction, then there exists an elliptic curve $\mathcal{E}$ defined over the integers $\mathbf{O}_K$ of K, whose generic fibre is E. We therefore see that:

$$E_4(\mathcal{E}) \in H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/\mathbf{O}_K})^{\otimes 4} \cong \mathbf{O}_K.$$

We now choose a fixed isomorphism $\alpha : H^0(E, \Omega^1_{E/K})^{\otimes 4} \to K$, such that the image, under the map α, of $H^0(\mathcal{E}, \Omega^1_{\mathcal{E}/\mathbf{O}_K})^{\otimes 4}$ is the ring of integers $\mathbf{O}_K$.

We now consider the isomorphisms from K to itself, which are given by elements θ of K^*. We are considering K as an abstract group here, rather than as a field. These act by multiplication: $\theta : x \mapsto \theta \cdot x$. Therefore, the image of $\mathbf{O}_K$ under θ is $\theta \cdot \mathbf{O}_K$, which is equal to $\mathbf{O}_K$ if and only if $v(\theta) = 0$.

We see therefore that although $E_4(E)$ is not well-defined the *valuation* $v(E_4(E))$ is. Therefore, we can talk about $v(E_4(E))$ without having to choose a specific isomorphism α.

Definition 2.8 (Coleman [11], page 448) *Consider $X_1(2^m)_{/\mathbf{Q}_2}$ as a rigid analytic space.*
If t is a point of $X_1(2^m)$ which corresponds to a cusp, then we define $v(E_4(t)) = 0$, following [6], section 4.
We define the ordinary locus of $X_1(2^m)$ to be the set of points t of $X_1(2^m)$ such that $v_2(E_4(t)) = 0$.
We define $Z_1(2^m)$ to be the rigid connected component of the ordinary locus in $X_1(2^m)$ which contains the cusp ∞. It is a rigid analytic space.
There is a natural extension of the line bundle ω to $Z_1(2^m)$, which we will also call ω.

In [12], page 36, it is shown that $Z_1(2^m)$ is an affinoid subdomain of the rigid space $X_1(2^m)_{/\mathbf{Q}_2}$.

We will perform calculations in later sections on the modular curve $X_0(2^m)$, which we will now define.

Definition 2.9 *Consider $X_1(2^m)$ as a modular curve. We see that the group $G := (\mathbf{Z}/2^m\mathbf{Z})^\times$ acts upon the non-cuspidal points of $X_1(2^m)$, by the following action: if $a \in (\mathbf{Z}/2^m\mathbf{Z})^\times$, then the action of a sends the pair (E, P) to (E, aP). This action extends to the cuspidal points of $X_1(2^m)$, and it sends cusps to cusps.*
We will define the modular curve $X_0(2^m)_{/\mathbf{Q}_2}$ to be the quotient of $X_1(2^m)$ by $(\mathbf{Z}/2^m\mathbf{Z})^\times$.

We note that the action of the group G does not change the valuation of a given elliptic curve E. We define $Z_0(2^m)$ to be the rigid connected component of the ordinary locus in $X_0(2^m)$ which contains the cusp ∞. It is a rigid analytic space.
We will now define strict affinoid neighbourhoods of $Z_0(2^m)$.

Definition 2.10 (Coleman [11], Section B2) *We think of $X_0(2^m)$ as a rigid space over $\mathbf{Q}_2$, and we let $t \in X_0(2^m)(\overline{\mathbf{Q}}_2)$ be a point, corresponding either to an elliptic curve defined over a finite extension of $\mathbf{Q}_2$, or to a cusp.*
Let w be a rational number, such that $0 < w < 2^{2-m}/3$, and $w < 1/4$.
We define $Z_0(2^m)(w)$ to be the connected component of the affinoid

$$\{t \in X_0(2^m) : \ v_2(E_4(t)) \leq 4w\}.$$

which contains the cusp ∞.

The condition involves $4w$ rather than w because we are working with a lifting of the fourth power of the Hasse invariant.
We are now ready to define w-overconvergent modular forms of weight 0.

Definition 2.11 (Coleman, [10], page 397) *Let w be a rational number, such that $0 < w < 2^{2-m}/3$ and $w < 1/4$. Let $\mathcal{O}$ be the structure sheaf of $Z_0(2^m)(w)$.*
We call sections of $\mathcal{O}$ on $Z_0(2^m)(w)$ w-overconvergent 2-adic modular forms of weight 0 and level $\Gamma_0(2^m)$.
If a section f of $\mathcal{O}$ is a w-overconvergent modular form, then we say that f is an overconvergent 2-adic modular form.
Let K be a complete subfield of $\mathbf{C}_2$, and define $Z_0(2^m)(w)_{/K}$ to be the affinoid over K induced from $Z_0(2^m)(w)$ by base change from $\mathbf{Q}_2$. The space $M_0(2^m, w; K) := \mathcal{O}(Z_0(2^m)(w)_{/K})$ of w-overconvergent modular forms of weight 0 and level $\Gamma_0(2^m)$ is a K-Banach space.

We define the q-expansion of an overconvergent modular form in a similar way to that of a modular form.

Definition 2.12 *Let f be an overconvergent modular form of level $\Gamma_0(2^m)$ and weight k. We define the q-expansion of f at ∞ to be the Laurent series obtained by evaluating f at the Tate curve.*

We finally define overconvergent modular forms with weight-character $\mathbf{k}$.

Definition 2.13 (Coleman-Mazur [12], page 46) *Let χ be a primitive Dirichlet character of conductor 2^m. Let $\mathbf{k} = (k, \chi)$ be a weight-character, and let K be a complete subfield of $\mathbf{C}_2$. Let w be a rational number, such that $0 < w < 2^{2-m}/3$ and $w < 1/4$. We say that*

$$F(q) = \sum_{n=0}^{\infty} a_n q^n \in K[[q]]$$

is the q-expansion of a w-overconvergent modular form of weight-character $\mathbf{k}$ defined over K, if $F(q)/E^*_{\mathbf{k}}$ is the q-expansion of a w-overconvergent modular form of weight 0 and level $\Gamma_0(4)$.

Let $\mathbf{k} = (k, \chi)$ be a weight-character, with the conductor of χ equal to 2^m, and let K be a complete subfield of $\mathbf{C}_2$. Following Coleman [10], page 397, we see that there is an injective map ϕ from the space of modular forms of weight-character $\mathbf{k}$ and tame level 1 with coefficients in K into the space of overconvergent modular forms with weight-character $\mathbf{k}$ and coefficients in K.

We will now show that certain spaces of overconvergent modular forms are in fact Banach spaces of the form $K\langle z\rangle$; that is to say, Tate algebras in one variable, where z is a carefully chosen modular function.

We have defined a valuation on the points t of the rigid space $X_0(2^m)$, based on the lifting of the Eisenstein series E_4. We recall that $j = E_4^3/\Delta$. Therefore, we see that, if the elliptic curve corresponding to t has good reduction, then $\Delta(t)$ has valuation 0, and therefore that

$$\begin{aligned} v(t) &= \frac{1}{4} v(E_4(t)) \\ &= \frac{1}{12} v((E_4)^3(t)) \\ &= \frac{1}{12} v(j(t)). \end{aligned}$$

We note that as E_4 is a lifting of A^4, and that any another lifting of A^4 would be of the form E_4+2F, where F is a modular form, then this valuation is well-defined if $0 \leq v(E_4(t)) < 1$. This corresponds to $0 \leq w < 1/4$.

We will now perform explicit calculations on $X_0(4)$ and $X_0(8)$.

Let $w \in \mathbf{Q}$ be in the region $0 \leq w < 1/4$.

From Lemma 2.2 of [16], we see that there is a uniformiser on the modular curve $X_0(4)$, which we will call j_4. It has the q-expansion

$$j_4 = \frac{1}{q\prod_{n=1}^{\infty}(1+q^n)^8(1+q^{2n})^8} = ((\Delta(q)/\Delta(q^4))^{1/3},$$

and the property that $j_4(\infty) = \infty$. We recall that $X_0(4)$ has genus 0.
Using the formulae on Chapter 2, page 19 of [16], we see that

$$\spadesuit : \; j = \frac{(j_4^2 + 256j_4 + 4096)^3}{j_4^4(j_4+16)}.$$

We recall that $Z_0(4)(w)$ is defined to be the connected component of

$$\{x \in X_0(4) : \; v(j(x)) \leq 12w\}$$

which contains the cusp ∞.
Therefore, we can rewrite the condition in the definition of $Z_0(4)(w)$ to be

$$v\left(\frac{(j_4^2 + 256j_4 + 4096)^3}{j_4^4(j_4+16)}\right) \leq 12w.$$

Now, as $j_4(\infty) = \infty$, we see that the connected component of the ordinary locus which contains ∞ is of the form $v(j_4) < D$, for some rational number D.
We consider the region $v(j_4) < 4$, and look at the numerator and denominator of $\spadesuit$. We see that $v((j_4^2 + 256j_4 + 4096)^3) = v(j_4^6)$, as $v(256j_4) > v(j_4^2)$, and $v(4096) > v(j_4^2)$. We also see that the valuation of the denominator of $\spadesuit$ is $v(j_4^5)$, as the valuation of $v(j_4)$ is less than 4.
Therefore, on the region $v(j_4) < 4$, the valuation of $\spadesuit$ is equal to the valuation of j_4.
Therefore we have proved

Theorem 2.14 *Let w be a rational number, such that $0 < w < 1/4$. Then*

$$Z_0(4)(w) = \{x \in X_0(4) : \; v(j_4(x)) \leq 12w\}.$$

We see that this definition includes only the component of the ordinary locus which contains ∞.
Let $\mathbf{k} = (1, \tau)$, where τ is the primitive Dirichlet character of conductor 4. We now consider the function

$$z := \frac{E_{\mathbf{k}}^*/V_{\mathbf{k}}^* - 1}{2}$$

which is a modular function on $X_0(8)$, because $V_{\mathbf{k}}$ is a modular form of level 8.
We will now show that z is an overconvergent modular form of weight 0, by recalling a result of Emerton.

Lemma 2.15 (Emerton [16], Appendix, Lemma 3, and Proposition 3.17) *Let μ be a fixed element of $\mathbf{O}_2$ with valuation 2^{2-m}. Let r be an element of $\mathbf{O}_2$ with valuation $2^{3-m} \cdot 3/4$. Then $E^*_{\mathbf{k}}/V^*_{\mathbf{k}}$ is a rigid analytic function on the open disc $v(j) < v(r)$, and*

$$E^*_{\mathbf{k}}/V^*_{\mathbf{k}} \in 1 + (\mu^2/j) \cdot \mathbf{O}_2[[\mu^{3/2}/j]].$$

Using this lemma, we see that

$$z \in (\mu/j) \cdot \mathbf{O}_2[[\mu^{3/2}/j]],$$

by re-arranging the formula in the Lemma, and then recalling the definition of z.
From Lemma 2.3 of [16], we see that there is a modular function j_8 which is a uniformiser on $X_0(8)$. It has q-expansion

$$j_8 = \frac{1}{q\prod_{n=1}^{\infty}(1+q^n)^4(1+q^{2n})^2(1+q^{4n})^4} = (\Delta(q)\Delta(q^4)/\Delta(q^2)\Delta(q^8))^{1/12}.$$

Also, $j_8(\infty) = \infty$.
On page 24 of [16], we see that

$$E^*_{\mathbf{k}}/V^*_{\mathbf{k}} = \frac{j_8+8}{j_8+4},$$

and therefore that

$$z = \frac{2}{j_8+4}.$$

Combining this with the formula relating j_4 and j_8 on page 19 of [16]:

$$\frac{1}{j_4} = \frac{1}{j_8} + \frac{4}{j_8^2},$$

we obtain a formula for j_4 in terms of z:

$$j_4 = \frac{8z^2 - 8z + 2}{z}.$$

Let w be a rational number such that $0 < w < 1/6$. We can define $Z_0(8)(w)$ by

$$Z_0(8)(w) = \left\{ x \in X_0(8) : \ v\left(\frac{8z^2-8z+2}{z}\right) \leq 12w \right\}.$$

Because we know that $j_4(\infty) = \infty$, we see from the formula in terms of z that j_4 is equal to ∞ when z takes the value 0.

We notice that this is isomorphic to $Z_0(4)(w)$; they are both regions of a modular curve of genus 0 such that $v(j_4) < 12w$.

We consider the region $v(z) > -1$, and look at $t := (8z^2 - 8z + 2)/z$. We see that the valuation of the numerator is 1 in this region, and therefore that the valuation of t is $1 - v(z)$.

We now choose $w = 1/12$.

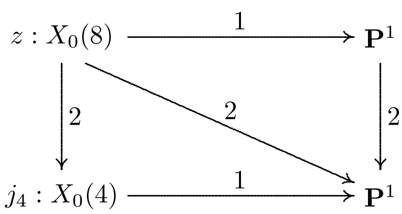
We see that the region $Z_0(8)(1/12)$ is given by

$$Z_0(8)(1/12) = \{x \in X_0(8) : \ v(z(x)) \geq 0\},$$

because the condition is $v(t) \leq 12w$, which is equivalent to $1 - v(z) \leq 1$ which is $v(z) \geq 0$.

Therefore we have shown that z is an isomorphism between $X_0(8)$ and $\mathbf{P}^1$.

Therefore we have the following commutative diagram (the numbers are the degrees of the maps between the objects):



Let D be the unit disc in $\mathbf{P}^1$ which is the image of $Z_0(4)(1/12)$. From the diagram above, there is an isomorphism between D and $z^{-1}(D) = Z_0(8)(1/12)$.

Now, the rigid functions on the closed disc over $\mathbf{Q}_2$ with centre 0 and radius 1 are defined to be power series of the form

$$\sum_{n \in \mathbf{N}} a_n z^n \ : \ a_n \in \mathbf{Q}_2, \ a_n \to 0.$$

Therefore, the 1/12-overconvergent modular forms of level $\Gamma_0(4)$ and weight 0 are

$$\mathbf{Q}_2 \langle z \rangle.$$

We now consider overconvergent modular forms of level 8. We define

$$j_{16} = \frac{1}{q \prod_{n=1}^{\infty} (1+q^n)^2 (1+q^{2n})(1+q^{4n})(1+q^{8n})^2} = \left(\frac{\Delta(q^{16})^2 \Delta(q^2)}{\Delta(q^8)\Delta(q)^2} \right)^{1/24}.$$

This is a uniformiser on $X_0(16)$, which has genus 0, and we see that

$$\frac{1}{j_8} = \frac{1}{j_{16}} + \frac{2}{j_{16}^2},$$

by multiplying out both sides to obtain an equality of modular forms of level 16. We also see that $j_{16}(\infty) = \infty$, by evaluating the above identity at ∞.

We also note that if $v(j_{16}) < 1$, then $v(j_{16}) = v(j_8)$, by considering the formula relating j_{16} and j_8. Let χ be the odd character of conductor 8, and let $\mathbf{k} = (1, \chi)$. We choose a square root of 2, and define

$$z_{16} = \frac{E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1}{\sqrt{2}}.$$

This is an overconvergent modular form of weight 0 and level 16, by Lemma 2.15. Let w be a rational number, such that $0 < w < 1/12$. By definition, we see that $Z_0(16)(w)$ is the connected component of the affinoid

$$\{t \in X_0(16): \ v(j(t)) \leq 12w\}$$

which contains the cusp ∞.

By combining the formulae relating j_{16}, j_8, j_4 and j, we see that

$$j = \frac{(j_{16}^8 + 256j_{16}^7 + 5632j_{16}^6 + 53248j_{16}^5 + 282624j_{16}^4 + 917504j_{16}^3 + 1835008j_{16}^2 + 2097152j_{16} + 1048576)^3}{(j_{16}^4 + 16j_{16}^3 + 96j_{16}^2 + 256j_{16} + 256)j_{16}^{16}(j_{16}^2 + 4j_{16} + 8)(j_{16} + 2)}.$$

We recall that $j(\infty) = j_{16}(\infty) = \infty$, and consider the region $v(j_{16}) < 1$. We see that, in this region, the valuation of the expression above is equal to that of j_{16}, because the valuation of the numerator is equal to $24v(j_{16})$, and the valuation of the denominator is equal to $23v(j_{16})$.

Therefore, we see that

$$Z_0(16)(w) = \{t \in X_0(16): \ v(j_{16}(t)) \leq 12w\}.$$

We notice that $Z_0(8)(w)$ and $Z_0(16)(w)$ are isomorphic, because both regions are given by $v(j_8) \leq 12w$, as $v(j_{16}) = v(j_8)$ if $v(j_8) < 1$.

We will now rewrite this definition in terms of z_{16}, which will allow us to show that z_{16} is an isomorphism between $X_0(16)$ and $\mathbf{P}^1$. We know that $X_0(16) \cong \mathbf{P}^1$, as the modular curve has genus zero, so there does exist an isomorphism between the two curves.

We note that

$$z_{16} = \frac{\sqrt{2}}{j_{16} + 2}.$$

This can be verified by multiplying both sides out, and proving the resulting identity of modular forms.

Now, if $v(j_{16}) < 1$, then we see that $v(z_{16}) = 1/2 - v(j_{16})$, so therefore

$$Z_0(16)(w) = \{t \in X_0(16) : \ v(z_{16}(t)) \geq 1/2 - 12w\}.$$

We now set $w = 1/24$, and see that

$$Z_0(16)(1/24) = \{t \in X_0(16) : \ v(z_{16}(t)) \geq 0\}.$$

As before, we have a commutative diagram: (where the numbers are the degrees of the maps between the objects)

$$\begin{array}{ccc} z : X_0(16) & \xrightarrow{\ 1\ } & \mathbf{P}^1 \\ \downarrow 2 & \searrow^{2} & \downarrow 2 \\ j_8 : X_0(8) & \xrightarrow{\ 1\ } & \mathbf{P}^1 \end{array}$$

Let D be the unit disc in $\mathbf{P}^1$ which is the image of $Z_0(8)(1/24)$. From the diagram above, there is an isomorphism between D and $z^{-1}(D) = Z_0(16)(1/24)$.

As before, we recall that the rigid functions on the closed unit disc over $\mathbf{Q}_2(\mu_4)$ with centre 0 and radius 1 are of the form

$$\sum_{n \in \mathbf{N}} a_n z_{16}^n \ : \ a_n \in \mathbf{Q}_2(\mu_4), \ a_n \to 0.$$

Therefore, the 1/24-overconvergent modular forms of level $\Gamma_0(8)$ and weight 0 are

$$\mathbf{Q}_2(\mu_4)\langle z \rangle.$$

We have defined z_{16} in terms of the Eisenstein series of weight-character $(1, \chi)$. Let χ' be the even primitive Dirichlet character of conductor 8, and let us define z'_{16} in terms of the Eisenstein series of weight-character $(2, \chi')$. We then notice that

$$z_{16} = z'_{16}$$

so therefore our calculations concerning overconvergent modular forms of level 8 are independent of our choice of weight 1 or weight 2 forms.

Remark 2.16 *The methods that we have used to show that these spaces of overconvergent modular forms are Tate algebras in one variable are clearly dependent on the fact that uniformisers are available for the modular curves of genus 0. We note that the genus of $X_0(32)$ is 1, so these* *methods will need modification, but it is possible that a suitably modified method could be used to show that certain overconvergent modular forms of higher level are also Tate algebras in one variable.*

Theorem 2.17 (Coleman [10], Theorem 1.1) *Every* 2*-adic overconvergent modular form of weight* k *with slope strictly less than* $k-1$ *is in the image of the space of classical modular forms of weight* k *under the map* ϕ.

2.3 The characteristic power series of an infinite matrix M

We will first define infinite matrices.

Definition 2.18 *Let* R *be a ring.*
An infinite matrix $M=(m_{i,j})$ *with coefficients in* R *is a two-dimensional array of elements* $M_{i,j}$ *of* R*, indexed by positive integers* i *and* j*. We say that* M *is* defined over R.
If n *is a positive integer, we define* M_n *to be the* $n \times n$ *matrix given by*

$$(M_n)_{i,j} = m_{i,j},\ \textit{for}\ 1 \le i,j \le n.$$

If the ring R *is a finite extension of* $\mathbf{Q}_2$*, with the nontrivial valuation* v_2 *defined on it, then we say that* M *is* compact *if, for all* $0 < N \in \mathbf{Q}$*, there exists a positive integer* a *such that, for all positive integers* $i,j:\ i > a$*, the valuation of* $m_{i,j}$ *is greater than* N.

We now define the characteristic power series of a compact infinite matrix.

Definition 2.19 (Serre [31], Proposition 7) *Let* M *be either a compact infinite matrix defined over* R*, or a finite matrix with entries in* R.
For every positive integer n*, we define* S *to be an ordered list of* n *distinct positive integers,* σ *to be a permutation acting on* S*,* $\varepsilon(\sigma)$ *to be the signature of* σ*, and we define* c_S *to be* $\sum_\sigma \varepsilon(\sigma)M(S,\sigma)$*, where* $M(S,\sigma) := \Pi_{i\in S} M_{\sigma(i),i}$.
The characteristic power series of M *is given by:*

$$det(1-tM) := \sum_{n=0}^{\infty} c_n t^n,\ \text{with}\ c_n = (-1)^n \sum_{\#S=n} c_S.$$

We will now quote a Proposition of Serre [31], which will tell us that, if M is compact, the characteristic power series of the finite matrices M_n tend coefficientwise towards the characteristic power series of the infinite matrix M.

Proposition 2.20 (Serre [31], Proposition 7.c) *Let* M *be a compact infinite matrix. Let* n *be an integer, and* M_n *defined as above. Then* $det(1-tM_n)$ *converges coefficientwise to* $det(1-tM)$*, as* $n \to \infty$.

We see from this Proposition that in order to prove results about the valuations of the coefficients of $det(1 - tM)$, we can consider the valuations of the coefficients of $det(1 - tM_n)$.

Proposition 2.21 *Let M be an $N \times N$ matrix defined over a finite extension of $\mathbf{Q}_2$, and let $0 \neq r \in \mathbf{Q}$. Let $det(1 - tM) = \sum_{i=0}^{N} c_i t^i$.*
Assume that

1. *For all positive integers n such that $1 \leq n \leq N$, the valuation of $det(M_n)$ is $r \cdot \frac{n(n+1)}{2}$.*
2. *The valuation of elements in column j is at least $r \cdot j$.*

Then we have that, for all positive integers m such that $1 \leq m \leq N$, $v_2(c_m) = r \cdot \frac{m(m+1)}{2}$.

To prove this Proposition, we consider c_S for $S \neq \{1, 2, \ldots, n-1, n\} =: I$, and note that $\varepsilon(\sigma)$ is just ± 1 by the definition of signature. From the definition of $M(S, \sigma)$, we see that each element of the product is in column i for $i \in S$, so has valuation at least $r \cdot i$, so the valuation of $M(S, \sigma)$ is at least $r \cdot \sum_{i \in S} i$ which is strictly greater than $r \cdot n(n+1)/2$.
We now look at the valuation of c_I, and we see that

$$v_2(c_I) = v_2\left(\sum_\sigma \varepsilon_\sigma M(I, \sigma)\right) = v_2(det(M_n)) = r \cdot \frac{n(n+1)}{2}.$$

Therefore, we have proved the Proposition. ∎

We will now prove a well-known result about the invariance of the determinant of a finite matrix under a certain transformation of its coefficients.

Lemma 2.22 *Let $M = (m_{i,j})$ be an $N \times N$ matrix defined over a field K, and let $\lambda \in K^*$. Define $M' = (m'_{i,j})$ to be the matrix with entries $(m_{i,j} \cdot \lambda^{j-i})$.*
Then $det(M) = det(M')$.

We define the $N \times N$ diagonal matrix $D(\lambda)$ by $D(\lambda)_{i,j} = \delta_{i,j} \cdot \lambda^i$, where $\delta_{i,j}$ is the Kronecker delta. We will prove the Lemma by considering the product of matrices $M' := D(\lambda^{-1}) \cdot M \cdot D(\lambda)$. We define $B = D(\lambda^{-1}) \cdot M$. Then we have:

$$B_{i,j} := (D(\lambda^{-1}) \cdot M)_{i,j} = \sum_{k=1}^{N} \delta_{i,k} \lambda^{-i} \cdot m_{k,j} = \lambda^{-i} \cdot m_{i,j}$$

and then

$$(B \cdot D(\lambda))_{i,j} = \sum_{k=1}^{N} \lambda^{-i} \cdot m_{i,k} \cdot \delta_{k,j} \cdot \lambda^j = \lambda^{j-i} \cdot m_{i,j}.$$

Therefore $M' = D(\lambda^{-1}) \cdot M \cdot D(\lambda)$. We also see that

$$det(M) = det(D(\lambda^{-1}) \cdot M \cdot D(\lambda)) = det(M'),$$

and hence the Lemma is proved. ∎

We now define the Newton polygon of a characteristic power series:

Definition 2.23 ([7], Section 3) *Let M be a compact infinite matrix defined over a finite extension of $\mathbf{Q}_2$, and let $F := det(1 - tM) = \sum_{i=0}^{\infty} c_i t^i$ be its characteristic power series. Let D be the convex hull of the points $\{(i, v_2(c_i))\}_{i \in \mathbf{N} \cup \{0\}}$.*
The Newton polygon of F *is the lower slope of D.*

We see that if we have a compact infinite matrix M such that the characteristic power series of all of the matrices M_n satisfy the conditions of Propostion 2.21 for a positive rational number r, then the slopes of the Newton polygon for the characteristic power series of M will be $\{r \cdot i\}_{i \in \mathbf{N}}$.

2.4 The U operator on overconvergent 2-adic cusp forms

Because we will mainly be concerned with the prime 2 in this thesis, we will write U instead of U_2 from now on. We will also fix a positive integer $m \geq 2$.
We will now prove a proposition similar to that in Coleman [11], page 451, which will tell us that the characteristic power series of the U operator acting on w-overconvergent modular forms of weight-character $\mathbf{k}$ is independent of the choice of w, as long as w is within the bounds $0 < w < 2^{2-m}/3$ and $w < 1/4$. This will mean that we only have to prove our results for a single choice of w.
We note that on page 451 of Coleman [11] this theorem is proved for overconvergent modular forms of level $\Gamma_0(4)$.

Proposition 2.24 *Let χ be a character of conductor 2^m, and let k be an integer. Let $\mathbf{k} = (k, \chi)$ be a weight-character. Let w_1, w_2 be two rational numbers, such that $0 < w_1 < w_2 < 2^{2-m}/3$ and $w_1, w_2 < 1/4$. Let M_{w_i} be the matrix of U acting on w_i-overconvergent modular forms of weight-character $\mathbf{k}$ with regard to an orthonormal basis.*
Then the two characteristic power series $det(1 - tM_{w_1})$ and $det(1 - tM_{w_2})$ are the same.

To prove the Proposition, we will now prove a Lemma.

Lemma 2.25 *Let w be a rational number, such that $0 < w < min(1/4, 2^{1-m}/3)$, and let w' be a rational number such that $w < w' < 2w$. Let M_w be the matrix of U acting on w-overconvergent modular forms of weight-character $\mathbf{k}$ with regard to an orthonormal basis.*
Then the two characteristic power series $det(1 - tM_w)$ and $det(1 - tM_{w'})$ are the same.

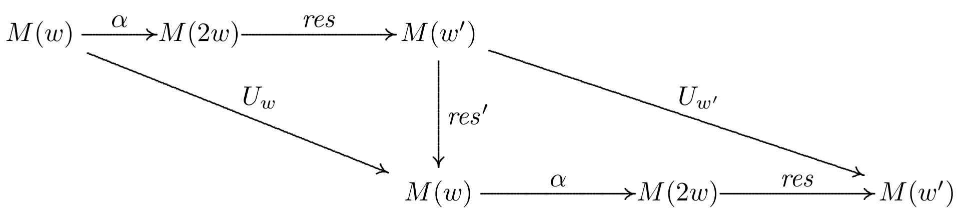
Let f be a w-overconvergent modular form, let E be an elliptic curve defined over a finite extension of $\mathbf{Q}_2$ and let C be a finite subgroup of E with order 2^m. The map α is defined to be

$$(\alpha f)(E, C) = \sum_{\substack{D \text{ order } 2 \\ D \not\subseteq C}} f(E/D, ImC).$$

We now quote a result of Buzzard.

Proposition 2.26 (Buzzard [6], Proposition 3.5) *Let E be an elliptic curve defined over a finite extension of $\mathbf{Q}_2$ with a finite subgroup C of order 2^m, such that $v(E) < 2^{2-m}/3$. Let D be a subgroup of E of order 2 such that $D \not\subseteq C$.*
Then $v(E/D) < 2^{1-m}/3$.

Using this proposition, we see that α is a map from w-overconvergent forms to $2w$-overconvergent forms. Now let w' be a rational number such that $w < w' < 2w$.
We define U_w to be the U operator acting on w-overconvergent modular forms and define $M(w)$ to be the space of w-overconvergent modular forms of weight 0 and level $\Gamma_1(2^m)$, and consider the following commutative diagram:



We now recall a Lemma of Serre:

Lemma 2.27 (Serre [31], Corollaire 2 of Proposition 7) *Let E and F be two Banach spaces. Let u be a continuous homomorphism from E to F and let v be a continuous homomorphism from F to E.*
If u is a compact ("completely continuous") homomorphism, then we have that

$$det(1 - t(u \circ v)) = det(1 - t(v \circ u)).$$

Now the map res' is compact, because it is a map of rigid spaces which is injective. We define $\beta = \alpha \circ res$. Therefore $\beta^{-1}U_{w'}$ is compact, because the diagram commutes.
We therefore have that $\beta^{-1}U_w\beta = U_{w'}$, and therefore that the two determinants $det(1 - tU_w)$ and $det(1 - tU_{w'})$ are the same, using the Lemma of Serre.
If $w_1 < 2^{1-m}/3$, then we can now find a sequence

$$w_1, w^{(1)}, w^{(2)}, \dots, w^{(s)}, w_2$$

such that $w_1 < w^{(1)} < 2w_1$, $w^{(i)} < w^{(i+1)} < 2w^{(i)}$, and $w^{(s)} < w_2 < 2w^{(s)}$. We then repeatedly apply Lemma 2.25 to prove the Proposition.
If $w_1 \geq 2^{1-m}/3$, then we apply the Lemma with $w = w_2/2$ and $w' = w_2$, then with $w = w_2/2$ and $w' = w_1$. This will prove the Proposition in this case also.
Therefore, we have shown that the characteristic polynomial of U is independent of choice of overconvergence parameter. ■

Once we have fixed a possible overconvergence parameter w, we can then talk about "the" characteristic polynomial of U acting on overconvergent modular forms without having to specify which value of the overconvergence parameter w we are dealing with.
We will now make an assumption about the valuation of the coefficients of the z-expansion of $U(z^2)$.

Assumption 2.28 *Let $\mathbf{k} = (k, \chi)$ be an integral weight-character, where χ has conductor exactly 2^m, and k is a positive integer.*
Let $z := (E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1)/\mu$.
We define $\{a_i\}$ by $U(z^2) = \sum_{i=1}^{\infty} a_i z^i$, and define $d := 2^{2-m}$.
Then we assume that

$$\diamondsuit : \quad v(a_i) = i \cdot d, \text{ for } i \text{ odd, and } v(a_i) > i \cdot d, \text{ for } i \text{ even.}$$

We note that we will be able to prove this assumption for several $\mathbf{k}$ in Chapter 3.

Definition 2.29 (Weight-character 0 Matrix) *Let $\mathbf{k} = (k, \chi)$ be a weight-character, with k a positive integer and χ a primitive Dirichlet character of conductor 2^m.*
We define the matrix $\mathbf{M}_{\mathbf{k},\mathbf{0}}$ to be $\mathbf{M}_{\mathbf{k},\mathbf{0}} := (m_{i,j})$, where $m_{i,j}$ is the coefficient of z^i in the z-expansion of $U(z^j)$. Also, we define the $n \times n$ matrix $\mathbf{M}_{\mathbf{k},\mathbf{0},n}$ by $(\mathbf{M}_{\mathbf{k},\mathbf{0},n})_{i,j} = m_{i,j}$.

We want to be able to consider the action of the U operator on overconvergent modular forms with non-zero weight-character $\mathbf{k}$. We will do this by using an observation from the work of Coleman which allows us to move between weight-character $\mathbf{0}$ and weight-character $\mathbf{k}$ via multiplication by a suitable quotient of modular forms.
From the discussion in Coleman [11], page 450, we see that the U operator acting on overconvergent modular forms of weight-character $\mathbf{k}$ has the same characteristic power series as the composition of the U operator acting on overconvergent modular forms of weight-character $\mathbf{0}$ with multiplication by the z-expansion of $E^*_{\mathbf{k}}/V^*_{\mathbf{k}}$.
We also recall that $E^*_{\mathbf{k}}/V^*_{\mathbf{k}} = 1 + \mu z$. Therefore, we can construct a new matrix which will have the same characteristic power series as the U operator in weight $\mathbf{k}$, by considering the matrix of the linear transformation $U \circ E^*_{\mathbf{k}}/V^*_{\mathbf{k}}$ from $K\langle z\rangle$ to $K\langle z\rangle$ relative to the basis $\{z^i\}$.

Definition 2.30 (Weight-character k Matrix) *Let $\mathbf{k} = (k, \chi)$ be a weight-character, with k a positive integer and χ a primitive Dirichlet character of conductor 2^m.*
We define $\mathbf{M}_{\mathbf{k}}$ to be the matrix given by $\mathbf{M}_{\mathbf{k}} := (m_{i,j})$, where $m_{i,j}$ is the coefficient of z^i in the z-expansion of $U(z^j) \cdot (1 + \mu z)$. We also define $\mathbf{M}_{\mathbf{k},n}$ to be the $n \times n$ matrix given by $(\mathbf{M}_{\mathbf{k},n})_{i,j} = (\mathbf{M}_{\mathbf{k}})_{i,j}$.

Let t be an integer, and let $\mathbf{k}' = ((2t+1)k, \chi^{2t+1})$ be an integral weight-character. We will define a matrix which has the same characteristic polynomial as the matrix of the U operator acting on modular forms of weight $\mathbf{k}'$, in terms of the basis $\{z^i\}$, which is defined in terms of the modular forms $E^*_{\mathbf{k}}$ and $V^*_{\mathbf{k}}$.

Using the observation from Coleman [11], page 450, we see that the U operator acting on overconvergent modular forms of weight-character $\mathbf{k}'$ has the same characteristic power series as the composition of the U operator acting on weight-character $\mathbf{0}$ overconvergent modular forms with multiplication by $(E^*_{\mathbf{k}}/V^*_{\mathbf{k}})^{2t+1}$.

We will therefore consider the matrix of the linear transformation $U \circ (E^*_{\mathbf{k}}/V^*_{\mathbf{k}})^{2t+1}$ from $K\langle z\rangle$ to $K\langle z\rangle$ relative to the basis $\{z^i\}$.

Definition 2.31 (Weight-character k′ Matrix) *Let $\mathbf{k} = (k, \chi)$ be a weight-character, with k a positive integer and χ a primitive Dirichlet character of conductor 2^m. Let t be an integer, and let $\mathbf{k}' = ((2t+1)k, \chi^{2t+1})$ be an integral weight-character.*

We define $\mathbf{M}_{\mathbf{k},\mathbf{k}'}$ to be the matrix given by $\mathbf{M}_{\mathbf{k},\mathbf{k}'} := (m_{i,j})$, where $m_{i,j}$ is the coefficient of z^i in the z-expansion of $U(z^j) \cdot (1+\mu z)^{2t+1}$. We also define $\mathbf{M}_{\mathbf{k},\mathbf{k}',n}$ to be the $n \times n$ matrix given by $(\mathbf{M}_{\mathbf{k},\mathbf{k}',n})_{i,j} = m_{i,j}$.

We will now set some notation on the values of $\mathbf{k}$ for the matrices $\mathbf{M}_{\mathbf{k}}$.

Notation 2.32 *When we refer to $\mathbf{M}_{\mathbf{k}}$ in future chapters, we will implicitly assume that $k \neq 0$. If we need to look at a weight 0 matrix, we will write it as $\mathbf{M}_{\mathbf{k},\mathbf{0}}$. We will also assume that χ is not the trivial character, except for the matrices $\mathbf{M}_{\mathbf{k},\mathbf{0}}$.*

2.5 The divisibility theorem

Let m be a fixed positive integer which is greater than 1. Let $\mathbf{k} = (k, \chi)$ be a weight-character, with k a positive integer and χ a primitive Dirichlet character of conductor 2^m.

First, we will show that the odd columns of the matrix $\mathbf{M}_{\mathbf{k}}$ are identically zero. Recall from Lemma 2.4 that the Eisenstein series $E^*_{\mathbf{k}}$ is an eigenvector with eigenvalue 1 for U, and that $U(V(E^*_{\mathbf{k}})) = E^*_{\mathbf{k}}$. Then we see that we have:

$$\begin{aligned} U(z) &= U\Big(\frac{E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1}{\mu}\Big) = \frac{1}{\mu} \cdot U\Big(\frac{E^*_{\mathbf{k}} - V^*_{\mathbf{k}}}{V^*_{\mathbf{k}}}\Big) \\ &= \frac{1}{\mu E^*_{\mathbf{k}}} \cdot U(E^*_{\mathbf{k}} - V^*_{\mathbf{k}}) = \frac{1}{\mu E^*_{\mathbf{k}}} \cdot (E^*_{\mathbf{k}} - E^*_{\mathbf{k}}) = 0. \end{aligned}$$

Hence we see that z has only odd q-coefficients, and that therefore z^2 has only even q-coefficients. Therefore z^{2i+1} has only odd q-coefficients. Hence for all non-negative integers t, we see that

$$U\big(z^{2t+1}\big) = 0.$$

Therefore, it suffices to consider even powers of z under the U operator.

We will also prove:

Lemma 2.33 *Let i be a positive integer. Then*

$$U\left(z^{2i}\right) = \left(U\left(z^2\right)\right)^i.$$

This will mean that we need only consider the action of the U operator on z^2, because we can calculate $U(z^{2i})$ from it.
Because z has only odd q-coefficients, we see that

$$z = qF(q^2) = qV(F(q)),$$

for some power series $F(q)$.
Therefore we have

$$U(z^{2i}) = U(q^{2i}V(F(q)^{2i})) = U(V(q^iF(q)^{2i})),$$

and hence we see that

$$U(z^{2i}) = q^iF(q)^{2i} = \left(qF(q)^2\right)^i = U(z^2)^i,$$

which proves the Lemma. ■

Theorem 2.34 (Main Theorem) *Let m be a positive integer greater than 1. Let $\mathbf{k} = (k, \chi)$ be an integral weight-character, where χ has conductor exactly 2^m, and k is a positive integer. Let t be an integer, and let $\mathbf{k}' = ((2t+1)\cdot k, \chi^{2t+1})$ be an integral weight-character. Let $d = 2^{2-m}$.*
Let $z = (E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1)/\mu$.
We define a_i by $U(z^2) = \sum_{i=1}^{\infty} a_i z^i$, and assume the criterion $\diamondsuit$ on the valuation of the a_i.
Let $w = 1/(2^m \cdot 3)$. We also assume that we have shown that:

$$\mathcal{O}(X_1(2^m)(w)) = K\langle z\rangle,$$

where K is the field over which μ is defined. Then:

1. *The slopes of overconvergent 2-adic cuspidal eigenforms of weight-character $\mathbf{k}$ are $\{2i \cdot d\}_{i\in\mathbf{N}}$.*

2. *The slopes of overconvergent 2-adic cuspidal eigenforms of weight-character $\mathbf{k}'$ are $\{2i \cdot d\}_{i\in\mathbf{N}}$.*

We recall that Theorem 2.17 tells us that overconvergent forms of weight k with slope less than $k-1$ are classical. We recall the following theorem from [9] on the dimensions of spaces of classical modular forms:

Theorem 2.35 (Cohen-Oesterlé [9], Théorème 1) *Let χ be a character of conductor 2^m, and let k be a positive integer. Let $\mathbf{k} = (k, \chi)$ be an integral weight-character.*
The dimension of the space of cuspidal modular forms of weight-character $\mathbf{k}$ is

$$2^{m-3}(k-1) - 1.$$

This will allow us to prove a corollary of the Main Theorem, on the slopes of classical modular forms. This will allow us to prove Theorem 1.1.

Corollary 2.36 *Let $\mathbf{k} = (k, \chi)$ be an integral weight-character, where χ has conductor 2^m and k is a positive integer. Let $c = 2^{3-m}$.*
Assume that we know that the criterion $\diamondsuit$ holds for overconvergent modular forms of weight-character $\mathbf{k}$.
The slopes of classical cuspidal modular forms of weight-character $\mathbf{k}$ are

$$c, 2c, \ldots, k-1-c.$$

The Corollary follows from Theorem 2.17, and the theorem of Cohen-Oesterlé, Theorem 2.35. We note that $k-1-c$ is strictly smaller than $k-1$, so that all of the forms are indeed classical, from the theorem of Coleman.
We will also prove a result about the ring over which the q-expansions at ∞ of the cusp forms of weight-character $\mathbf{k}$ are defined. This will allow us to prove Corollary 1.2.

Corollary 2.37 *Let $\mathbf{k} = (k, \chi)$ be an integral weight-character, where χ has conductor 2^m and k is a positive integer. Let K be the field $\mathbf{Q}_2(\mu)$.*
Assume that we know that the criterion $\diamondsuit$ holds for overconvergent modular forms of weight-character $\mathbf{k}$, and that the coefficients of the characteristic polynomial of the U operator are elements of K.
We also assume that the characteristic power series of $\mathbf{M_k}$ has coefficients in K.
Let $f = \sum_{n=1}^{\infty} a_n q^n$ be a classical cuspidal modular eigenform of weight-character $\mathbf{k}$.
Then $a_n \in K$ for all n.

First, we note that because we have made the assumption $\diamondsuit$ we can assume that the slopes of classical cuspidal modular forms of weight-character $\mathbf{k}$ are $c, 2c, \ldots, k-1-c$. In particular, they are all distinct.
We will recall a fact from Ribet [30], page 21, to prove this Corollary. Let σ be an element of $Gal(\overline{K}/K)$. Then we have that

$$\sigma(f) := \sum_{n=1}^{\infty} \sigma(a_n) q^n$$

is a classical cuspidal modular eigenform of weight-character $\mathbf{k}$.

We see that the valuation of $\sigma(a_2)$ is the same as that of a_2, because the characteristic polynomial of a_2 is stable under conjugation by σ. Therefore, $\sigma(f)$ is an eigenform of weight-character $\mathbf{k}$ with the same slope as f. Hence $\sigma(f) = f$, because there is only one classical eigenform of weight-character $\mathbf{k}$ which has any given slope, by Corollary 2.36.

This means that $\sigma(f) = f$ for all σ. Therefore $a_n \in K$ for all positive integers n. ■

In Section 2.6, we will prove Theorem 2.34, part 1, under the condition $\diamondsuit$. In Section 2.7 we will prove Theorem 2.34, part 2, again under the assumption $\diamondsuit$. Chapter 3 will be devoted to proving the statement $\diamondsuit$ for certain $\mathbf{k} = (k, \chi)$, which will prove the Main Theorem for all $\mathbf{k}' = ((2t+1)k, \chi^{2t+1})$.

We fix an integral weight-character $\mathbf{k}$, and consider the matrix $M := \mathbf{M_k} = (m_{i,j})$, under the assumption that we know $\diamondsuit$ for $\mathbf{k}$. We see that, for every positive integer n, we can define finite matrices

$$(M'_n)_{i,j} := \mu^{j-i} \cdot m_{i,j}.$$

We will now show that these matrices are defined over the same ring as $\mathbf{M_k}$.

The condition $\diamondsuit$ tells us that the valuations of the z-coefficients of $U(z^2) = \sum_{i=1}^{\infty} a_i z^i$ are

$$v(a_i) = i \cdot d, \text{ for } i \text{ odd, and } v(a_i) > i \cdot d, \text{ for } i \text{ even.}$$

The effect of performing the transformation from M_n to M'_n is to rewrite $\tau = \mu z$, and to consider the τ-coefficients of $U(\tau^2)$. We see that the valuations of the τ-coefficients of $U(\tau^2) = \sum_{i=1}^{\infty} b_i \tau^i$ are given by

$$\diamondsuit' : \ v(b_i) = 2^{3-m} \text{ for } i \text{ odd, and } v(b_i) > 2^{3-m}, \text{ for } i \text{ even.}$$

The coefficients of the $(2i)^{\text{th}}$ column of the matrix M'_n therefore have valuation at least $(2^{3-m})^i$, as they are defined by being the coefficients of τ^j in the expansion of $(U(\tau^2))^i$.

We emphasise that we are not changing the orthonormal basis of the space of overconvergent modular forms on which we calculate the matrix of the U operator. This manipulation is performed simply to enable us to apply Proposition 2.21. We will only be considering the characteristic polynomials of the *finite* matrices M'_n; we will then apply Lemma 2.22 and Proposition 2.20 to prove the Main Theorem.

We observe that the odd-numbered columns of M are identically zero, because $U(z^{2i+1}) = 0$. We will deal with this by considering the matrix N_n, defined by

$$(N_n)_{i,j} := (M'_{2n})_{2i,2j}, \text{ where } 1 \le i, j \le n.$$

This has the same characteristic power series as M_{2n}. We will seek to apply Proposition 2.21 to the matrices N_n. We note that the elements of the i^{th} column of N_n have valuation at least $2id$. Because we are interested only in the 2-valuation of the determinants of the matrices N_n, we will write each N_n as a product of a diagonal matrix D_n, and another matrix N'_n. We will then show that N'_n has unit determinant in $\mathbf{O}_2$, by considering its reduction modulo a prime ideal above 2. We will then be able to read off the determinant of N_n, by considering the diagonal matrix D_n.
We define the diagonal matrix D_n by

$$(D_n)_{i,j} = \delta_{i,j} \cdot \lambda^{-i}, \text{ for } 1 \leq i, j \leq n,$$

where $\delta_{i,j}$ is the Kronecker delta symbol. We see therefore that the valuation of the determinant of D_n is $d \cdot n(n+1)$.
We define N'_n by the formula

$$(N'_n)_{i,j} = \lambda^i \cdot (N_n)_{i,j}, \text{ where } 1 \leq i, j \leq n.$$

If we can prove that the valuation of the determinant of N'_n is 0, then we will have shown that the valuation of the determinant of N_n is $d \cdot n(n+1)$. We see that, because we know $\diamondsuit'$, we know that the elements of the i^{th} column of N_n have valuation at least $2id$. Therefore, the elements of the i^{th} column of N'_n have valuation at least 0, so are elements of the ring of integers of K.
We therefore consider the matrices $\overline{N}_n$ defined by

$$\left(\overline{N}_n\right)_{i,j} := (N'_n)_{i,j} \mod \mathfrak{p}$$

where $\mathfrak{p}$ is the maximal ideal of the field K. Because the elements of N'_n are elements of $\mathbf{O}_k$, we see that $\overline{N}_n$ is well-defined.
We now consider the z-expansion of $\overline{U(z^2)}$ with the coefficients of z^i reduced modulo the maximal ideal $\mathfrak{p}$ of K and we see that

$$\overline{U(z^2)} \equiv z + z^3 + z^5 + \cdots + z^{2i+1} + \cdots \equiv \frac{z}{1+z^2}.$$

We therefore see, using Lemma 2.33, that

$$\overline{U(z^{2i})} \equiv \left(\frac{z}{1+z^2}\right)^i \mod \mathfrak{p}.$$

We will introduce a notion that is extensively used by Smithline [32]; that of the *generating function* of a matrix. This will allow us to write the coefficients of $\overline{N}_n$ in an efficient way.

Definition 2.38 *The* generating function *F of an infinite matrix $M = (m_{i,j})$ defined over a* *ring R is the power series in two variables x and y*

$$F(x,y) := \sum_{i,j \in \mathbf{N}} m_{i,j} \cdot x^i y^j \in R[[x,y]].$$

We will seek to write our generating functions as rational functions in x and y wherever possible. Now, we can write down the generating function of the matrices $\overline{N}_n$ with weight-character $\mathbf{0}$ as a simple and explicit rational function, because we can write

$$F_{\mathbf{0}}(x,y) := \sum_{j=1}^{\infty} \left(\frac{xy^2}{1+x^2} \right)^j = \frac{\frac{xy^2}{1+x^2}}{1 - \frac{xy^2}{1+x^2}} = \frac{xy^2}{1+x^2+xy^2} \in \mathbf{F}_2[[x,y]].$$

This generating function is not quite what we want, however, because the $(i,j)^{\text{th}}$ entry of $\overline{N}_n$ is given by the coefficient of $x^{2i}y^{2j}$ in $F_{\mathbf{0}}(x,y)$. We will, in the next section, rewrite $F_{\mathbf{0}}(x,y)$ in a form which gives the $(i,j)^{\text{th}}$ entry of $\overline{N}_n$ as the coefficient of $x^i y^j$.

Definition 2.39 *Let $\mathbf{k} = (k,\chi)$ be a weight-character, with $k > 0$ and $\chi \neq \mathbf{1}$. Let $F_{\mathbf{0}}(x,y)$ be the generating function for the matrix $\overline{N}$ in weight-character $\mathbf{0}$. We define the generating function $F_{\mathbf{k}}(x,y)$ to be*

$$F_{\mathbf{k}}(x,y) := F_0(x,y) \cdot (1+x).$$

In the next section we will write $F_{\mathbf{k}}$ as a function of x and y, by algebraic manipulation, and then we will show that the matrices $\overline{N}_n$ defined by $F_{\mathbf{k}}$ all have unit determinant in $\mathbf{O}_2$.

2.6 The Matrix theorem

We will assume for the remainder of this chapter that we have proved the assumption $\diamondsuit$ for a given weight-character $\mathbf{k}$.

Let n be an positive integer. We will consider the matrix $\overline{N}_n$ that we defined above. We recall that this has generating function $F_{\mathbf{k}}(x,y)$.

We will now write the function $F_{\mathbf{k}}$ solely in terms of x^2 and y^2, which will allow us to compute the matrices $\overline{N}_n$.

We consider the generating function $F_{\mathbf{k}}$:

$$F_{\mathbf{k}}(x,y) = \frac{\frac{xy^2}{1+x^2}}{1 + \frac{xy^2}{1+x^2}} \cdot (1+x) = \left(\frac{xy^2}{1+x^2} + \left(\frac{xy^2}{1+x^2} \right)^2 + \cdots \right) \cdot (1+x),$$

and we see that the even terms are given by the sum of these expansions:

$$\frac{\left(\frac{xy^2}{1+x^2}\right)^2}{1+\left(\frac{xy^2}{1+x^2}\right)^2}+\frac{x\cdot\frac{xy^2}{1+x^2}}{1+\left(\frac{xy^2}{1+x^2}\right)^2}\in \mathbf{F}_2[[x,y]],$$

and after some algebra we find that this is equal to

$$F(x^2,y^2)=\frac{x^2y^2(1+x^2+y^2)}{1+x^4+x^2y^4}\in \mathbf{F}_2[[x,y]].$$

We replace x^2 by x and y^2 by y to obtain the generating function of a new matrix. This will enable us to compute the terms of the characteristic power series of $\overline{\mathbf{M}}_{\mathbf{k}}$:

$$\overline{F}_{\mathbf{k}}(x,y):=\frac{xy(1+x+y)}{1+x^2+xy^2}\in \mathbf{F}_2[[x,y]].$$

Definition 2.40 *Let k be a positive integer, and let χ be a character of conductor 2^m. Let $\mathbf{k}=(k,\chi)$ be a weight-character, and define $\overline{F}_{\mathbf{k}}(x,y)$ as above.*
We define the infinite matrix $\overline{M}_{\mathbf{k}}:=(m_{i,j})$, where $m_{i,j}$ is the coefficient of x^iy^j in the expansion of $\overline{F}_{\mathbf{k}}(x,y)$. We also define the matrix $\overline{M}_{\mathbf{k},n}$ to be the $n\times n$ matrix such that $(\overline{M}_{\mathbf{k},n})_{i,j}=(\overline{M}_{\mathbf{k}})_{i,j}$, for $1\le i,j\le n$.

We will suppress the $\mathbf{k}$ for the rest of this section, and write simply $\overline{M}$ and $\overline{M}_n$. We see that $\overline{M}_n=\overline{N}_n$. Now from the discussion after the Main Theorem (Theorem 2.34) it will suffice to show that each of the $\overline{M}_n$ has determinant of valuation 0.
We notice that there is only one term in the denominator of $\overline{F}$ with any power of y in it. This means that we can try to write $\overline{F}$ as a series of sums, one for each column, as y is the column variable.
We can split the generating function $\overline{F}$ easily into even and odd columns, as there is exactly one term in $\overline{F}$ which contains an odd power of y, and it is in the numerator. So

$$\overline{F}_{\text{odd}}(x,y):=\frac{xy(1+x)}{1+x^2+xy^2},\ \overline{F}_{\text{even}}(x,y):=\frac{xy^2}{1+x^2+xy^2}.$$

We now rebracket these as follows:

$$\overline{F}_{\text{odd}}(x,y):=\frac{xy}{(1+x)(1+\frac{xy^2}{(1+x)^2})},\ \overline{F}_{\text{even}}(x,y):=\frac{\frac{xy^2}{1+x^2}}{1+\frac{xy^2}{(1+x)^2}}.$$

We now use the binomial expansion to expand the fraction $1/(1+\frac{xy^2}{(1+x)^2})$ out (over $\mathbf{F}_2$), and we find that we have a series which explicitly gives the n^{th} column of the matrix $\overline{M}$, for any n. We see that the binomial expansions are, after being tidied up:

$$\overline{F}_{\text{odd}}(x,y) := \frac{xy}{1+x} + \frac{x^2y^3}{(1+x)^3} + \frac{x^3y^5}{(1+x)^5} + \cdots + \frac{xy(xy^2)^{n-1}}{(1+x)^{2n-1}} + \cdots$$

and

$$\overline{F}_{\text{even}}(x,y) := \frac{xy^2}{(1+x)^2} + \frac{x^2y^4}{(1+x)^4} + \cdots + \frac{(xy^2)^n}{(1+x)^{2n}} + \cdots$$

So we know what the general terms are in this sequence. To add two distinct columns together, we multiply the column with the lower power of y in its numerator by the power of y which will give them the same power of y in their numerators, then add the two generating functions together. For instance, to add column 1 to column 2:

$$y \cdot \frac{xy}{1+x} + \frac{xy^2}{(1+x)^2} = \frac{xy^2(1+x) + xy^2}{(1+x)^2} = x \cdot \frac{xy^2}{(1+x)^2},$$

and we notice that the result is the generating function for column 2 multiplied by x. In effect, by adding column 1 to column 2, we have moved it down one row in the matrix.
If we add the new column 2 to column 3, then we find that the same thing happens; the result is x times the generating function for column 3.
A natural next question is: will this always work? Can we always use this procedure to reduce the finite matrix $\overline{M}_n$ to a lower triangular matrix $\overline{M}'_n$ by elementary row and column operations, and use this to show that all the $\overline{M}_n$ all have determinant 1? If we know this then it will follow that the matrices M'_n will have determinants of valuation $c \cdot n(n+1)/2$, which will allow us to use Proposition 2.21 to prove that the slopes are $\{i \cdot c\}_{i \in \mathbf{N}}$.

Lemma 2.41 *Let n be a positive integer, and consider the matrix $\overline{M}_n$ that we defined above. We can transform $\overline{M}_n$ by Gaussian elimination into a lower triangular matrix $\overline{M}'_n$ with 1s on the diagonal, and hence show that it has determinant 1.*
The lower triangular matrix we obtain has generating function

$$F_{\text{LT}} := \frac{xy}{1+x+xy},$$

with coefficients in $\mathbf{F}_2$.

To prove this we define column operations that will reduce the matrix to a lower triangular form, and then prove that they do in fact do what they are supposed to.

Definition 2.42 *Let $\overline{M}_n$ be the $n \times n$ matrix truncation of $\overline{M}$ that we defined above. We define ψ_{2a+1} as the operation of adding column $2a+1$ of $\overline{M}_n$ to column $2a+2$ of $\overline{M}_n$, and ϕ_{2a} as the operation of adding column $2a$ of $\overline{M}_n$ to column $2a+1$ of $\overline{M}_n$.*

We define Ψ_{2a+1} *as being the operations* $\{\psi_{2a+1}, \psi_{2a+3}, \dots, \psi_{\lfloor \frac{n}{2} \rfloor -1}\}$ *carried out together and* Φ_{2a} *as being the operations* $\{\phi_{2a}, \phi_{2a+2}, \dots, \phi_{\lceil \frac{n}{2} \rceil -2}\}$ *carried out together.*

These operate in the following manner: the Ψ_a operators move the even columns which have numbers greater than or equal to $a+1$ down one row, while the Φ_a operators move the odd columns with numbers greater than or equal to $a+1$ down one row. Notice that this means that after the first a operators are applied to $\overline{M}_n$, the first a columns are left unchanged by the actions of the operators.

Algorithm 2.43 *To lower triangularise* $\overline{M}_n$ *we apply the operators as follows:*

$$\Psi_1, \Phi_2, \dots, \Phi_{2c}, \Psi_{2c+1}, \dots, \Phi_{2\lceil \frac{n}{2} \rceil -2}, \Psi_{2\lfloor \frac{n}{2} \rfloor -1}.$$

Here, $\lceil \alpha \rceil$ *and* $\lfloor \alpha \rfloor$ *are the ceiling and floor operators respectively, both acting on* α.

We now prove that the ψ_{2a+1} and ϕ_{2a} operators work as they are supposed to, which will be enough to prove the theorem, as the operators Ψ and Φ are built from ψ and ϕ, and the lower-case operators together act to move the matrix's columns down one row. This makes proving that the ψ and ϕ operators work more straightforward.

The ψ_{2a+1} operator adds column $2a+1$ to column $2a+2$. So we have (remembering that this is all being considered in characteristic 2):

$$\text{New Column } 2a+2 = \frac{(xy^2)^{a+1}}{(1+x)^{2a+2}} + \frac{y \cdot xy(xy^2)^a}{(1+x)^{2a+1}} = x \cdot \left(\frac{(xy^2)^{a+1}}{(1+x)^{2a+2}} \right),$$

which is what we wanted to prove.

And the ϕ_{2a} operator adds the shifted-down column $2a+2$ to column $2a+3$, so we have again:

$$\text{New Column } 2a+3 = \frac{y \cdot x(xy^2)^{a+1}}{(1+x)^{2a+2}} + \frac{xy(xy^2)^{a+1}}{(1+x)^{2a+3}} = x \cdot \left(\frac{xy(xy^2)^{a+1}}{(1+x)^{2a+3}} \right).$$

So we have shown that the two lowercase operators do both work, hence the uppercase operators work also, so we have shown that this does actually follow through. Also, we notice that all of these operations are reversible, as we are in characteristic 2, so adding a column twice is the same as not adding it at all.

Now, as the n^{th} column has leading coefficient $x^n y^n$, we see that the n^{th} column of the matrix $\overline{M}'$ is

$$\left(\frac{xy}{1+x} \right)^n,$$

and hence, by summing these, we find that the generating function for $\overline{M}'$ is

$$F_{\mathrm{LT}} = \frac{\frac{xy}{1+x}}{1 + \frac{xy}{1+x}} = \frac{xy}{1 + x + xy},$$

as required. ■

So we have proved that the determinant of each of the $\overline{M}_n$ is a unit in $\mathbf{O}_2$. Now, using Proposition 2.21, and the discussion after the statement of the Main Theorem, we see that this, together with the fact that the infinite matrix $\overline{M}$ is being postmultiplied by the diagonal matrix D implies that the valuation of the n^{th} term of the characteristic polynomial is $c \cdot n$.

We can now see that the slopes are $\{i \cdot c\}$, so we are done.

This proves the Main Theorem for weight-character $\mathbf{k}$. ■

In the next section we will use this result to prove part 2 of the Main Theorem.

2.7 More general weights

Now that we have proved some useful facts about weight-character $\mathbf{k}$, we look at what can be done for more general integer weight-characters.

In this section, we will in fact prove:

Theorem 2.44 (Main Theorem, Part 2) *We assume that we have proved the Main Theorem, part 1, for a weight-character $\mathbf{k} = (k, \chi)$, and that we know the assumption $\diamondsuit$.*

Let $c := 2^{3-m}$, where 2^m is the conductor of χ.

Then the slopes for weight-character $\mathbf{k}' = (k(2t+1), \chi^{2t+1})$, for any integer t, are

$$\{i \cdot c\}_{i \in \mathbf{N}}\,.$$

We will use the modular function $(E^*_{\mathbf{k}}/V^*_{\mathbf{k}})^{2t+1}$ as the multiplier to turn our weight 0 matrix to a weight $\mathbf{k}'$ matrix, using Definition 2.31.

We will show that the formula for the generating function mod 2 of the matrix in weight-character $\mathbf{k}'$ is

$$F_{\mathbf{k},\mathbf{k}'}(x,y) = \frac{xy(1+x+y)\cdot(1+x)^t}{1+x^2+xy^2},$$

and then we will consider what the extra factor of $(1+x)^t$ does to the matrix, and how much the arguments used above still work.

In fact, we notice that the lowest power of x in the expansion is not touched, as there is a 1 in the new multiplier. So in fact the first term of the column expansion given above is not moved at all, so our results on the determinants of the matrices $\overline{M}_{\mathbf{k},n}$ will carry over, and we will be able to conclude that they have determinants of valuation $c, 2c, \ldots, c \cdot n, \ldots$, which is what we set out to do.

We need to show that $F_{\mathbf{k},\mathbf{k}'} = (1+x)^t \cdot F_{\mathbf{k}}$, to extend the result from the last section to more general weights.

From above, we note that the generating function for the matrix $\overline{M}_{\mathbf{k},\mathbf{k}'}$ is given as $F_{\mathbf{k},\mathbf{k}'}(x,y) := (xy^2(1+x)\cdot(1+x)^{2t})/(1+x^2+xy^2)$. We need to find out what the terms in this with even powers of x are.

We see that $F_{\mathbf{k},\mathbf{k}'}(x,y) = F_{\mathbf{k}}(x,y)\cdot(1+x)^{2t}$, and furthermore that $(x+1)^{2t+2} = (1+x^2)^t$, so therefore the terms of $F_{\mathbf{k},\mathbf{k}'}(x,y)$ with even powers of x are just the even terms of $F_{\mathbf{k}}$ multiplied by $(1+x^2)^t$.

Therefore, we have that the terms with even powers of x are given by:

$$F_{\mathbf{k},\mathbf{k}'}(x,y) = \frac{x^2y^2(1+x^2+y^2)}{1+x^4+x^2y^4}\cdot(1+x^2)^t$$

and we replace x^2 with x and y^2 with y to obtain the generating function $\overline{F}_{\mathbf{k}'}$:

$$\overline{F}_{\mathbf{k},\mathbf{k}'}(x,y) = \frac{xy(1+x+y)}{1+x^2+xy^2}\cdot(1+x)^t = \overline{F}(x,y)\cdot(1+x)^t,$$

which is what we set out to prove. Also, notice that we can perform Ψ and Φ as above, as there is a constant multiplier $(x+1)^t$ for both the odd and even columns, so we can just take the multiplier $(x+1)^t$ outside the brackets and use the algorithm detailed above to reduce $\overline{M}_{\mathbf{k},\mathbf{k}',n}$ to a lower triangular matrix, and the proof that this works in weight $\mathbf{k}$ will still be valid for the more general weights $\mathbf{k}'$.

The lower triangular matrix that we obtain using the Ψ and Φ operators has generating function

$$F_{\mathbf{k}',\,\mathrm{LT}} = \frac{xy\cdot(1+x)^t}{1+x+xy}.$$

Therefore, we have proved part 2 of the Main Theorem, under the assumption $\diamondsuit$. ■

Chapter 3

Proofs for levels 4 and 8

We have proved in the last chapter that if we know the divisibility result $\diamondsuit$ for the coefficients of the z-expansion of $U(z^2)$ then we can in fact show that the slopes are $\{c \cdot i\}_{i \in \mathbf{N}}$. We will now prove this divisibility result for level 4 and all odd weights, level 8 and all odd weights, and level 8 and all even weights congruent to 2 mod 4.

This will also prove the theorems that we stated in the introductory section, Theorem 1.1 and Corollary 1.2.

Having proved theorem 2.6 and theorem 2.5, to show that the slopes are $\{c \cdot i\}_{i \in \mathbf{N}}$ we need to show the divisibility result $\diamondsuit$ from the Main Theorem, Theorem 2.34, for the coefficients of z in the expansion of $U(z^2)$. We will do these by showing explicitly that $U(z^2)$ is a rational function of z.

First, we note that

$$U(z^2) = U\left(\frac{(E_{\mathbf{k}}^*)^2 - 2E_{\mathbf{k}}^* V_{\mathbf{k}}^* + (V_{\mathbf{k}}^*)^2}{\mu^2 (V_{\mathbf{k}}^*)^2}\right) = \frac{1}{(\mu E_{\mathbf{k}}^*)^2} U((E_{\mathbf{k}}^*)^2 - 2E_{\mathbf{k}}^* V_{\mathbf{k}}^* + (V_{\mathbf{k}}^*)^2).$$

We also recall a theorem from Agashe and Stein [1], which applies a result of Sturm [33] to obtain a bound on the number of Hecke operators needed to generate a Hecke algebra:

Theorem 3.1 ("Sturm bound", Sturm, [33]) *The ring of Hecke operators acting on the space of modular forms of weight k and level N with coefficients in a subring R of $\mathbb{C}$ is generated by the Hecke operators T_n, with*

$$n \leq \frac{kN}{12} \cdot \prod_{p|N} \left(1 + \frac{1}{p}\right).$$

3.1 Level 4, odd weight.

Theorem 3.2 *Let t be an integer, and let τ be the odd character of conductor 4.*

The slopes of the U operator acting on overconvergent 2-adic cuspidal eigenforms of weight-character $\mathbf{k} = (2t+1, \tau)$ are $\{2i\}_{i \in \mathbf{N}}$.
If f is a normalised classical cuspidal eigenform of weight-character $\mathbf{k}$, then the Fourier coefficients of $f(q)$ are elements of $\mathbf{Q}_2$.

We prove this by verifying the divisibility result $\diamondsuit$.
From Definition 2.1, we see that

$$\begin{aligned} B_{0,\tau} + B_{1,\tau} \cdot t + B_{2,\tau} \cdot \frac{t^2}{2!} + \cdots &= \sum_{a=1}^{4} \frac{\tau(a) \cdot t \cdot \mathbf{e}^{at}}{\mathbf{e}^{4t} - 1} \\ &= \frac{t\mathbf{e}^{t} - t\mathbf{e}^{3t}}{\mathbf{e}^{4t} - 1} \\ &= -\frac{t}{2} + \cdots \end{aligned}$$

By comparing the coefficients of t, we see that $B_{1,\tau} = -1/2$. Therefore, $\lambda = \frac{-B_{1,\tau}}{2 \times 1} = 1/4$, and hence μ is equal to ± 2, by Definition 2.2. We choose the positive square root, and we see that, for $\mathbf{k} = (1, \tau)$, $z = (E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1)/2$.
We will now prove the divisibility result.

Proposition 3.3 *Let τ be the nontrivial character of conductor 4, and let $\mathbf{k} = (1, \tau)$. Define $z = (E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1)/2$.*
Then we have that:

$$U(z^2) = \frac{2z}{(1+2z)^2} = \frac{E^*_{\mathbf{k}} V^*_{\mathbf{k}} - (V^*_{\mathbf{k}})^2}{(E^*_{\mathbf{k}})^2}.$$

We have chosen $\mu = 2$, and we see that $U(z^2)$ is independent of our choice, because the square root is absorbed in the square.
If we had chosen -2, then we would have

$$U(z^2) = \frac{-2z}{(1+2z)^2}.$$

The valuations of each term of this z-expansion are the same as those of the z-expansion in the Proposition. This means that the Proposition is independent of the choice of μ.
We notice that to show this equality, it will suffice to show that

$$U((E^*_{\mathbf{k}})^2 - 2E^*_{\mathbf{k}} V^*_{\mathbf{k}} + (V^*_{\mathbf{k}})^2) = 4(E^*_{\mathbf{k}} V^*_{\mathbf{k}} - (V^*_{\mathbf{k}})^2).$$

Because both sides of this are modular forms of weight 2 and level 8, we can apply the Sturm bound criterion (see [1]), which says that if the first few coefficients of two modular forms are the same, then they are in fact equal. We note that in this case the Sturm bound is 2, and hence the two are in fact identical: we have

$$U((E_{\mathbf{k}}^*)^2 - 2E_{\mathbf{k}}^*V_{\mathbf{k}}^* + (V_{\mathbf{k}}^*)^2) = 4(E_{\mathbf{k}}^*V_{\mathbf{k}}^* - (V_{\mathbf{k}}^*)^2) = q + 4q^3 + 6q^5 + 13q^9 + 12q^{11} + \cdots$$

We then notice that the valuation of the i^{th} coefficient in the z-expansion of $U(z^2)$ is i if i is odd, and greater than i if i is even, so the divisibility result is shown.
We then use the Main Theorem and Corollary 2.37, with this divisibility result, to prove the Theorem. ■

3.2 Level 8, odd weight.

Theorem 3.4 *Let t be an integer, and let χ be the odd character of conductor 8.*
The slopes of overconvergent 2-adic cuspidal eigenforms of weight-character $\mathbf{k} = (2t+1, \chi)$ are $\{i\}_{i\in\mathbf{N}}$.
If f is a normalised classical cuspidal eigenform of weight-character $\mathbf{k}$, then the Fourier coefficients of $f(q)$ are elements of $\mathbf{Q}_2$.

We prove this by verifying the divisibility result, using a direct computation of $U(z^2)$.
From Definition 2.1, we see that

$$\begin{aligned} B_{0,\chi} + B_{1,\chi}\cdot t + B_{2,\chi}\cdot\frac{t^2}{2!}\cdots &= \sum_{a=1}^{8}\frac{\chi(a)\cdot t\cdot \mathbf{e}^{at}}{\mathbf{e}^{8t}-1} \\ &= \frac{t(\mathbf{e}^{t}+\mathbf{e}^{3t}-\mathbf{e}^{5t}-\mathbf{e}^{7t})}{\mathbf{e}^{8t}-1} \\ &= -t+\cdots \end{aligned}$$

By comparison of the coefficients of t, we can calculate the Bernoulli number $B_{1,\chi}$ to be -1. Therefore $\lambda = -\frac{B_{1,\chi}}{2\times 1} = \frac{1}{2}$, and hence μ is a square root of 2. We see that, for $\mathbf{k} = (1,\chi)$, $z = (E_{\mathbf{k}}^*/V_{\mathbf{k}}^* - 1)/\sqrt{2}$. We choose and fix a square root of 2 in $\mathbf{Q}_2(\mu_4)$.
We will now prove the assumption $\diamondsuit$; we will in fact show that $U(z^2)$ is a rational function of z.

Proposition 3.5 *Let χ be the odd character of conductor 8, and let $\mathbf{k} = (1,\chi)$. Define $z = (E_{\mathbf{k}}^*/V_{\mathbf{k}}^* - 1)/\sqrt{2}$.*
Then:

$$U(z^2) = \frac{\sqrt{2}z}{1+2z^2} = \frac{V_{\mathbf{k}}^*E_{\mathbf{k}}^* - (V_{\mathbf{k}}^*)^2}{(E_{\mathbf{k}}^*)^2 - 2E_{\mathbf{k}}^*V_{\mathbf{k}}^* + 2(V_{\mathbf{k}}^*)^2}.$$

We see that $U(z^2)$ is independent of our choice of a square root of 2, and that the z-expansion coefficients of the two possibilities for $U(z^2)$ differ only by a unit. Therefore, our choice of a square root of 2 does not affect the Proposition.

To prove the equality in the Proposition, it will suffice to prove that

$$U((E_{\mathbf{k}}^*)^2 - 2E_{\mathbf{k}}^*V_{\mathbf{k}}^* + (V_{\mathbf{k}}^*)^2) \cdot ((E_{\mathbf{k}}^*)^2 - 2E_{\mathbf{k}}^*V_{\mathbf{k}}^* + 2(V_{\mathbf{k}}^*)^2) = 2(E_{\mathbf{k}}^*)^2(E_{\mathbf{k}}^*V_{\mathbf{k}}^* - (V_{\mathbf{k}}^*)^2),$$

which can be verified by checking that the first few coefficients on both sides are the same:

$$\text{LHS} = \text{RHS} = \frac{1}{4}q + q^2 + 3q^3 + 8q^4 + 31/2q^5 + 28q^6 + 46q^7 + 64q^8 + 373/4q^9 \cdots,$$

then invoking [1], as the Sturm bound here is 8. We can therefore apply the Main Theorem and Corollary 2.37. ■

3.3 Level 8, weight $k \equiv 2 \mod 8$.

Theorem 3.6 *Let t be an integer, and let χ be the even character of conductor 8.*
The slopes of overconvergent 2-adic cuspidal eigenforms of weight-character $\mathbf{k} = (4t+2, \chi)$ are $\{i\}_{i \in \mathbf{N}}$.
If f is a normalised classical cuspidal eigenform of weight-character $\mathbf{k}$, then the Fourier coefficients of $f(q)$ are elements of $\mathbf{Q}_2$.

Again, we will prove this by verifying the divisibility result.

From Definition 2.1, we see that

$$\begin{aligned} B_{0,\chi} + B_{1,\chi} \cdot t + B_{2,\chi} \cdot \frac{t^2}{2} + \cdots &= \sum_{a=1}^{8} \frac{\chi(a) \cdot t \cdot \mathbf{e}^{at}}{\mathbf{e}^{8t} - 1} \\ &= \frac{t(\mathbf{e}^t - \mathbf{e}^{3t} - \mathbf{e}^{5t} + \mathbf{e}^{7t})}{\mathbf{e}^{8t} - 1} \\ &= t^2 + \cdots \end{aligned}$$

We can calculate the Bernoulli number $B_{2,\chi}$ to be 2, by comparison of the coefficients of t^2. Therefore $\lambda = -\frac{B_{2,\chi}}{2 \times 2} = -\frac{1}{2}$, and so μ is a square root of -2.

We see that, for $\mathbf{k} = (2, \chi)$, $z = (E_{\mathbf{k}}^*/V_{\mathbf{k}}^* - 1)/\sqrt{-2}$. We choose and fix a square root of -2 in $\mathbf{Q}_2(\mu_4)$.

By explicit computation of $U(z^2)$, we will now show that $U(z^2)$ is a rational function of z.

Proposition 3.7 *Let χ be the even character of conductor 8, and let $\mathbf{k} = (2, \chi)$. Let $z = (E_{\mathbf{k}}^*/V_{\mathbf{k}}^* - 1)/(\sqrt{-2})$.*

Then we have:

$$U(z^2) = \frac{\sqrt{-2}z}{1-2z^2} = \frac{E^*_{\mathbf{k}}V^*_{\mathbf{k}} - (V^*_{\mathbf{k}})^2}{(E^*_{\mathbf{k}})^2 - 2E^*_{\mathbf{k}}V^*_{\mathbf{k}} + 2(V^*_{\mathbf{k}})^2}.$$

We see that $U(z^2)$ is independent of our choice of square root of -2, and that the z-expansion coefficients of the two possibilities for $U(z^2)$ differ only by a unit. Therefore, our choice of square root of -2 does not affect the Proposition.

We can verify that the equality in the Proposition holds by checking that

$$U((E^*_{\mathbf{k}})^2 - 2E^*_{\mathbf{k}}V^*_{\mathbf{k}} + (V^*_{\mathbf{k}})^2) \cdot (2(E^*_{\mathbf{k}})^2 - 4E^*_{\mathbf{k}}V^*_{\mathbf{k}} + 4(V^*_{\mathbf{k}})^2) = 4(E^*_{\mathbf{k}})^2((V^*_{\mathbf{k}})^2 - E^*_{\mathbf{k}}V^*_{\mathbf{k}}),$$

which we do, as before, by checking that the coefficients of both sides are the same up to q^i for the i that we will need when applying the Sturm bound, which is 16. We see that both sides are in fact equal to

$$\frac{1}{2}q - 2q^2 - 2q^3 + 16q^4 - 9q^5 - 24q^6 + 44q^7 - 128q^8 + \frac{133}{2}q^9 + 420q^{10}$$
$$-438q^{11} + 192q^{12} - 237q^{13} - 2032q^{14} + 2596q^{15} + 1024q^{16} + \cdots,$$

and we then invoke [1] again. We can therefore apply the Main Theorem and Corollary 2.37. ■

Chapter 4

Further Study

In the previous chapters, we have considered overconvergent 2-adic modular forms with weight-character. A natural question is: can we prove similar results to the Main Theorem for other primes, or are our results for $p = 2$ just a numerical coincidence tied in to the fact that 2 is a very small prime indeed?

We present here some numerical results, and record some conjectural formulae for the i^{th} slope of overconvergent p-adic modular forms of weight-character $\mathbf{k} = (k, \chi)$, where k is a nonzero integer and χ is a primitive Dirichlet character of conductor p^2, such that $\chi(-1) = (-1)^k$.

4.1 $p = 3$

We used the MAGMA package (see [2]) to compute the spaces of classical modular forms of weight-character $\mathbf{k} = (k, \chi)$ and tame level 1 for $2 \leq k \leq 32$, for an odd or even primitive Dirichlet character χ of conductor 9, with $\chi(-1) = (-1)^k$.

Having considered the slopes of U_3 acting on these spaces of modular forms, we make the following conjecture.

Conjecture 4.1 *Let k be a nonzero integer and let χ be a primitive Dirichlet character of conductor 9, such that $\chi(-1) = (-1)^k$. Let $\mathbf{k} = (k, \chi)$ be an integral weight-character.*

The (3-)slopes of overconvergent 3-adic cuspidal modular forms of weight-character $\mathbf{k}$ and tame level 1 are:

$$\{i\}_{i \in \mathbf{N}} .$$

Jacobs [22] has considered the action of the U_3 operator on spaces of modular forms over certain quarternion algebras and has proved the following theorem, using similar techniques to those used in previous chapters of this thesis:

Theorem 4.2 (Jacobs) *The slopes of U_3 acting on the spaces of 2-new cusp forms of level 18 and even weight k which are not integers are of the form*

$$\left\{\frac{2t-1}{2}\right\}_{t\in\mathbf{N}}.$$

Each of these slopes appears with multiplicity 2.

For bigger primes, the situation is not quite so simple. The valuations are not given by a simple linear function, but they do seem closely related to one.

4.2 $p = 5$

There seem to be two distinct patterns of slopes here, closely related to one another. We consider k even, and $k \leq 14$.

Let χ be a primitive Dirichlet character of conductor 25, such that $\chi(-1) = 1$, and let $\mathbf{k} = (k, \chi)$ be a weight-character.

If $k \equiv 2 \mod 6$, then the slopes of overconvergent 5-adic modular forms of weight-character $\mathbf{k}$ and tame level 1 are in the sequence

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots$$

while if $k \not\equiv 2 \mod 6$, the slopes of overconvergent 5-adic modular forms of weight-character $\mathbf{k}$ and tame level 1 are in the sequence

$$2, 4, 5, 7, 8, 10, 12, 13, 15, 16, 18, 20, 21, 23, 24, 26, \ldots$$

We see that there are "jumps" at three points modulo 5 in the second of these sequences. At the $(5n+1)^{th}$, $(5n+2)^{th}$ and $(5n+4)^{th}$ slopes, there is a difference of 2, whereas for the others there is a difference of 1. Motivated by the example of Buzzard and Calegari [8], we have conjectural formulae:

Conjecture 4.3 *Let k be an even integer, and let χ be a primitive Dirichlet character of conductor 25, such that $\chi(-1) = 1$. Let $\mathbf{k} = (k, \chi)$ be a weight-character.*

Let n be a positive integer.

If $k \not\equiv 2 \mod 6$, then the n^{th} slope of overconvergent 5-adic cuspidal modular forms of weight-character $\mathbf{k}$ and tame level 1 is given by:

$$s_{k \bmod 6}(n) := n + \lfloor\frac{n+1}{5}\rfloor + \lfloor\frac{n+3}{5}\rfloor + \lfloor\frac{n+4}{5}\rfloor.$$

If $k \equiv 2 \mod 6$, then the n^{th} slope of overconvergent 5-adic cuspidal modular forms of weight-character $\mathbf{k}$ *and tame level 1 is given by:*

$$s_2(n) := n + \lfloor \frac{n}{5} \rfloor + \lfloor \frac{n+1}{5} \rfloor + \lfloor \frac{n+3}{5} \rfloor.$$

We also see that there is a relation between these two formulae; indeed we have that

$$s_4(n) = s_0(n) = s_2(n+3) - 4, \text{ for } n \geq 0.$$

4.3 $p = 11$

We will not make any conjectures for the slopes of overconvergent 11-adic modular forms, as we do not have enough evidence. However, we will mention a conjecture of Calegari [8], which gives a formula similar to those that we have conjectured for overconvergent 5-adic modular forms.

Conjecture 4.4 (Calegari) *Let $p = 11$ and $\mathbf{k} = (0, \mathbf{1})$. Then the n^{th} slope of overconvergent 11-adic modular forms of weight-character* $\mathbf{k}$ *and tame level 1 is given by*

$$v_{11}\left(\frac{\lfloor \frac{6n+1}{5} \rfloor ! \lfloor \frac{6n+4}{5} \rfloor !}{\lfloor \frac{n}{5} \rfloor !^2}\right) + \sum_{k=1}^{4} \left\lfloor \frac{n+k}{5} \right\rfloor,$$

where v_{11} is the 11*-adic valuation.*

4.4 Overview

Let p be a fixed prime, and let $\mathbf{k} = (k, \chi)$ be a weight-character, with k a nonzero integer and χ a primitive Dirichlet character of conductor p^2, such that $\chi(-1) = (-1)^k$.
Let $E^*_{\mathbf{k}}$ be an Eisenstein series of weight-character $\mathbf{k}$, with a nonzero constant term. Using the definition given in Miyake [27], we see that such a series exists. We define $V^*_{\mathbf{k}} = V(E^*_{\mathbf{k}})$.
We define a quotient of modular forms x by:

$$x := \lambda \cdot (E^*_{\mathbf{k}}/V^*_{\mathbf{k}} - 1).$$

We then define $z = \mu x$, where μ is a fixed element of $\mathbf{Q}_p(\zeta_{p^2})$ such that $\mu^2 = \frac{1}{\lambda}$.
The general plan would be to use Coleman's definition of the matrix $M_{\mathbf{k}}$ to create a matrix representing the action of the U_p operator on powers of z^i, and then to use techniques like those of Chapter 2 and Chapter 3 to give information on the slopes of U_p, by considering the characteristic power series of $M_{\mathbf{k}}$. Hopefully, a criterion could be found similar which is similar to the condition $\diamondsuit$ in the Main Theorem, using information on the p-valuations of the z-expansion of $U_p(z^i)$ to prove

results on the slopes of overconvergent modular forms, using the characteristic power series result of Serre [31].

Some refinements and modifications may be necessary in the proof of the Main Theorem to take care of the fact that when the residue characteristic of $\mathbf{Q}_p$ is not 2, then $1+1 \neq 0$, but this seems less important than being able to prove facts about the z-expansion of $U_p(z^i)$, in the manner of Chapter 3.

Part II

Explicit calculations with Hecke algebras

Chapter 5

Some non-Gorenstein Hecke algebras attached to spaces of modular forms

5.1 Introduction

In this chapter we exhibit some examples of non-Gorenstein Hecke algebras, and hence some modular forms for which mod 2 multiplicity one does not hold.

Define $S_2(\Gamma_0(N))$ to be the space of classical cuspidal modular forms of weight 2, level N, and trivial character. The Hecke algebra $\mathbf{T}_N$ is defined to be the subring of $\mathrm{End}(S_2(\Gamma_0(N))$ generated by the Hecke operators $\{T_p : p \not| N\}$ and $\{U_q : q|N\}$. Let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}_N$, and let ℓ denote the characteristic of the finite field $\mathbf{T}_N/\mathfrak{m}$. By work of Shimura, one can associate to $\mathfrak{m}$ a semi-simple Galois representation $\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{T}_N/\mathfrak{m})$ satisfying $\mathrm{tr}(\rho(\mathrm{Frob}_p)) \equiv T_p \bmod \mathfrak{m}$ for all primes $p \nmid N\ell$. We say that $\mathfrak{m}$ is non-Eisenstein if $\rho_{\mathfrak{m}}$ is absolutely irreducible.

As an example, if E is a (modular) elliptic curve over $\mathbf{Q}$ of conductor N, let $f := \sum_{n\geq 1} a_n q^n$ be the modular form in $S_2(\Gamma_0(N))$ associated to E. Associated to f is a minimal prime ideal of $\mathbf{T}_N$; we say that $\mathfrak{m}$ is associated to f, or to E, if $\mathfrak{m}$ contains this minimal prime ideal. In this case, the representation associated to $\mathfrak{m}$ is isomorphic to the semisimplification of $E[\ell]$, where ℓ is the characteristic of $\mathbf{T}_N/\mathfrak{m}$.

We say that a local ring R is *Gorenstein* if R is isomorphic to $Hom_{\mathbf{Z}_p}(R, \mathbf{Z}_p)$, and that a ring is *Gorenstein* if all of its localisations at maximal ideals are Gorenstein.

The localisation $(\mathbf{T}_N)_{\mathfrak{m}}$ of $\mathbf{T}_N$ at a maximal ideal $\mathfrak{m}$ is frequently a Gorenstein ring, and such a phenomenon is related to the study of the $\mathfrak{m}$-torsion in the Jacobian $J_0(N)$ of $X_0(N)$. For example, if $\mathfrak{m}$ is non-Eisenstein then by the main result of [3], the $\mathfrak{m}$-torsion in $J_0(N)$ is isomorphic to a direct sum of $d \geq 1$ copies of $\rho_{\mathfrak{m}}$. If $d = 1$ then one says that the ideal $\mathfrak{m}$ satisfies "mod ℓ multiplicity one", or just "multiplicity one". In this case, the localisation $(\mathbf{T}_N)_{\mathfrak{m}}$ is known to be Gorenstein. Multiplicity one is a common phenomenon for maximal ideals $\mathfrak{m}$ of $\mathbf{T}_N$. Let us restrict for the rest of this chapter to the case of non-Eisenstein maximal ideals $\mathfrak{m}$. The first serious study of these mod ℓ multiplicity one questions is that of Mazur [25], who proves that if $N = q$ is prime and the characteristic of the finite field $\mathbf{T}_q/\mathfrak{m}$ is not 2 then $\mathfrak{m}$ satisfies multiplicity one and hence $(\mathbf{T}_q)_{\mathfrak{m}}$ is Gorenstein. He also showed that, if $\mathbf{T}_q/\mathfrak{m}$ has characteristic 2 and $T_2 \in \mathfrak{m}$ then $(\mathbf{T}_q)_{\mathfrak{m}}$ is Gorenstein. This work has been generalised by several authors, to higher weight cases and non-prime level. Rather than explaining these generalisations in complete generality, we summarise what their implications are in the case left open by Mazur.

We fix notation first. From now on, the level $N = q$ is prime, $\mathfrak{m}$ is a non-Eisenstein maximal ideal of $\mathbf{T}_q$, and the characteristic of $\mathbf{T}_q/\mathfrak{m}$ is 2. Let $f = \sum_n a_n q^n$ be the mod 2 modular form associated to f.

As we said already, if $a_2 = 0$ then Mazur proved that $\mathfrak{m}$ satisfies multiplicity one. Gross proves in [20] that if $a_2 \neq 1$ then $\mathfrak{m}$ satisfies multiplicity one, and in Chapter 12 of [20] (see the top of page 494) he states as an open problem whether multiplicity one holds in the remaining case $a_2 = 1$.

Edixhoven [15] proves that if $\rho_{\mathfrak{m}}$ is ramified at 2, then $\mathfrak{m}$ satisfies multiplicity 1 (Theorem 9.2, part 3). Buzzard [5] and [4] shows that if $\rho_{\mathfrak{m}}$ is unramified at 2 but if the image of Frobenius at 2 is not contained in the scalars of $\mathrm{GL}_2(\mathbf{T}_q/\mathfrak{m})$ then $\mathfrak{m}$ satisfies multiplicity one; see Proposition 2.4 of [5].

The main theorem of this chapter is that, for $q \in \{431, 503, 2089\}$ (note that all of these are prime), there is a non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbf{T}_q$ lying above 2, such that $(\mathbf{T}_q)_{\mathfrak{m}}$ is not Gorenstein. As a consequence, these $\mathfrak{m}$ do not satisfy mod 2 multiplicity one. In fact, the theorems above served as a very useful guide to where to search for such $\mathfrak{m}$.

It is proved in Matsumura [24], Theorem 21.3, that if a $\mathbf{Z}_2$-algebra is finitely generated and a complete intersection, then it is Gorenstein. Hence the $\mathbf{T}_q$ above are not complete intersections. The work of Wiles [37] and Taylor-Wiles [34] on Fermat's Last Theorem proves as a byproduct that certain Hecke algebras are complete intersections. Hence this chapter gives a bound on how effective Wiles and Taylor's methods can be in characteristic 2.

Each of the maximal ideals $\mathfrak{m} \subset \mathbf{T}_q$ that we construct are non-Eisenstein and have the property that $\mathbf{T}_q/\mathfrak{m} = \mathbf{F}_2$, so let us first consider a general surjective representation $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{F}_2)$. For such a representation, the trace of ρ at any unramified prime p can be computed if one knows the splitting of p in the extension of $\mathbf{Q}$ of degree 6 cut out by ρ. Let K denote this extension. If p splits completely then Frob_p has order 1 and if p splits into 3 primes in K, then Frob_p has order 2. In both cases the trace of $\rho(\mathrm{Frob}_p)$ is 0. The other possibility is that p splits into two primes, and then Frob_p has order 3 and the trace of $\rho(\mathrm{Frob}_p)$ is 1.

By the theorems of Mazur, Gross, Edixhoven and Buzzard above, if we wish to find examples of maximal ideals where multiplicity one fails, we could adopt the following approach: we firstly search for modular elliptic curves E of prime conductor q, such that 2 is unramified in $K = \mathbf{Q}(E[2])$, the field generated over $\mathbf{Q}$ by the coordinates of the 2-torsion of E. We then require that 2 splits completely in K and that K has degree 6 over $\mathbf{Q}$. Note that these conditions imply that E has good ordinary reduction at 2. For any such elliptic curves that one may find, the maximal ideals of $\mathbf{T}_q$ associated to $E[2]$ will be maximal ideals not covered by any of the multiplicity one results above.

We search for curves like this by using a computer to search pre-compiled tables, such as Cremona [14], and compute the associated maximal ideals of $\mathbf{T}_q$. For each such maximal ideal, we then explicitly construct the completion of $(\mathbf{T}_q)_\mathfrak{m}$ as a subring of a direct sum of finite extensions of $\mathbf{Q}_2$ and check to see whether it is Gorenstein. Note that a Noetherian local ring is Gorenstein if and only if its completion is. Hence this procedure will test the Gorenstein-ness of localisations of $\mathbf{T}_q$ at non-Eisenstein maximal ideals not covered by the above theorems.

It is slightly surprising to note that after trying only one or two examples, one finds maximal ideals where multiplicity one fails. Our discoveries are summarised in the following theorem.

Theorem 5.1 *Assume that $q \in \{431, 503, 2089\}$. Then there is a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_q$ such that $(\mathbf{T}_q)_\mathfrak{m}$ is not Gorenstein, and hence for which mod 2 multiplicity one does not hold.*

Note that as a consequence, the ring $\mathbf{T}_q$ is in these cases, by definition, not Gorenstein.

The calculations behind this theorem were made possible with the HECKE package, running on the MAGMA computer algebra system [2]. MAGMA includes an environment for specialised number-theoretic calculations, and also incorporates Cremona's database of elliptic curves, and the HECKE package, written by William Stein, implements efficient algorithms to generate spaces of modular forms of arbitrary level, weight (≥ 2) and character over global and finite fields. Without this computing package this chapter could not have been written.

An application of the main result of this chapter can be found in Section 6 of Emerton [17]. Let X be the free $\mathbf{Z}$-module of divisors supported on the set of singular points of the curve $X_0(q)$ in characteristic q. Theorem 0.5 of [17] shows that $\mathbf{T}_q$ is Gorenstein if and only if X is an invertible $\mathbf{T}_q$-module. The module X can be explicitly calculated quickly using the Mestre-Oesterle method of graphs, from [26], as implemented in, e.g., MAGMA.

The exact sequence of $\mathrm{Gal}(\overline{K}/K)$-modules on page 488 of Gross [20], is split as an exact sequence of Hecke modules, in the Gorenstein case. Emerton proves that the analogue of this short exact sequence in the non-Gorenstein case is never split. He uses this to prove results about the $\mathfrak{m}$-adic Tate module of $J_0(q)$.

In Ribet-Stein [29] the existence of non-Gorenstein Hecke algebras is discussed in the context of the level optimisation procedure associated with Serre's conjecture (see section 3.7.1).

5.2 $\mathbf{T}_{431}$

Theorem 5.2 *There is a maximal ideal* $\mathfrak{m}$ *of* $\mathbf{T}_{431}$ *such that* $(\mathbf{T}_{431})_{\mathfrak{m}}$ *is not Gorenstein.*

Set $q = 431$. There are two non-isogenous modular elliptic curves E_1 and E_2 of conductor q, defined over $\mathbf{Q}$, with corresponding minimal Weierstrass equations

$$E_1: \; y^2 + xy = x^3 - 1$$
$$E_2: \; y^2 + xy + y = x^3 - x^2 - 9x - 8.$$

The corresponding modular forms $d_1, d_2 \in S_2(\Gamma_0(431))$ have Fourier coefficients in $\mathbf{Z}$. One can easily check that the fields $\mathbf{Q}(E_i[2])$ are isomorphic, and of degree 6 over $\mathbf{Q}$, which shows that the two elliptic curves have isomorphic and irreducible mod 2 Galois representations. One checks that 2 splits completely in $K = \mathbf{Q}(E_i[2])$.
Let

$$\rho : \mathrm{Gal}(K/\mathbf{Q}) \longrightarrow \mathrm{Aut}(E_1[2])$$

denote the associated mod 2 Galois representation. We claim that there are precisely four eigenforms in $S_2(\Gamma_0(431))$ giving rise to ρ. To see this, note firstly that $\rho(\mathrm{Frob}_3)$ has order 3 and hence has trace 1. We compute the characteristic polynomial of T_3 acting on $S_2(\Gamma_0(431))$ and reduce it modulo 2. We find that the resulting polynomial is of the form $(X-1)^4 g(X)$, where $g(1) \neq 0$. This shows already that there are at most four eigenforms which could give rise to ρ. We now compute the q-expansions of these four candidate eigenforms, as elements of $\overline{\mathbf{Q}}_2[\![q]\!]$. Two of the eigenforms, say d_1 and d_2, corresponding to the elliptic curves E_1 and E_2 above, of course have coefficients in $\mathbf{Z}$. The other two, d_3 and d_4, are conjugate and are defined over $\mathbf{Q}_2(\sqrt{10})$. We now check that the reductions to $\mathbf{F}_2$ of the first 72 q-expansion coefficients (the Sturm bound) of all four of these eigenforms are equal. By [33], this implies that the four forms themselves are congruent, and hence all give rise to isomorphic mod 2 Galois representations. By construction, these representations must all be isomorphic to ρ.
Let $\mathfrak{m}$ denote the corresponding maximal ideal of $\mathbf{T}_q$. Our goal now is to compute the completion $\mathbf{T}$ of $(\mathbf{T}_q)_{\mathfrak{m}}$ explicitly. Let V be the 4-dimensional space over $\overline{\mathbf{Q}}_2$ spanned by the four eigenforms giving rise to ρ. The ring $\mathbf{T}$ is the $\mathbf{Z}_2$-subalgebra of $\mathrm{End}(V)$ generated by the Hecke operators T_p for $p \neq 431$, and U_{431}. By the theory of the Sturm bound, this algebra equals the $\mathbf{Z}_2$-algebra generated by T_n for $n \leq 72$. One readily computes this algebra. Let α be the coefficient of q^3 in d_3. Then d_3 and d_4 are defined over $\mathbf{Q}_2(\alpha) = \mathbf{Q}_2(\sqrt{10})$. It turns out that $\mathbf{T}$ is isomorphic to the subring of $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2[\alpha]$ generated as a $\mathbf{Z}_2$-module by

$$\{(1,1,1), (0,2,0), (0,0,\alpha+1), (0,0,2)\}.$$

This is a local ring, with unique maximal ideal $\mathfrak{m}$ generated as a $\mathbf{Z}_2$-module by

$$\{(2,0,0),(0,2,0),(0,0,\alpha+1),(0,0,2)\}.$$

Now we claim that there is a reducible parameter ideal (recall that a parameter ideal is an ideal that contains a power of the maximal ideal). This will show that $\mathbf{T}$ is not Gorenstein (see [24], Theorem 18.1). (Note that $\mathbf{T}$ is Cohen-Macaulay, as it has Krull dimension one and has no nonzero nilpotent elements (see [24], Section 17, page 139)).

The ideal $\mathfrak{i}$ generated by $(2,2,\alpha+1)$ is a parameter ideal, since it contains $\mathfrak{m}^2$. We observe that the ideals

$$\mathfrak{i}_1 = ((2,0,0),(0,2,\alpha+1)) \quad \text{and} \quad \mathfrak{i}_2 = ((0,2,0),(2,0,\alpha+1))$$

have intersection exactly $\mathfrak{i}$. Hence $\mathfrak{i}$ is a reducible parameter ideal, so $\mathbf{T}$ is not Gorenstein, and as $\mathbf{T}$ is the completion of the Noetherian local ring $(\mathbf{T}_{431})_{\mathfrak{m}}$ we deduce that this ring is also not Gorenstein. Hence mod 2 multiplicity one fails for $\mathfrak{m}$.

5.3 $\mathbf{T}_{503}$

Theorem 5.3 *There is a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_{503}$ such that $(\mathbf{T}_{503})_{\mathfrak{m}}$ is not Gorenstein.*

The argument is similar to that of the previous section, and we merely sketch it. In fact, the case $q = 503$ is slightly technically simpler than the case of $q = 431$ because all four relevant newforms are defined over $\mathbf{Z}_2$.

There are three isogeny classes of modular elliptic curves of conductor 503. Let F_1, F_2 and F_3 denote representatives in each class, and let f_1, f_2 and f_3 denote the corresponding modular forms. One checks that the fields $\mathbf{Q}(F_i[2])$ generated by the 2-torsion of the three elliptic curves are all isomorphic. If K denotes this extension, then one checks that K has degree 6 over $\mathbf{Q}$, and that 2 is unramified and splits completely in the integers of K. Let $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{F}_2)$ denote the corresponding Galois representation, and let $\mathfrak{m}$ denote the maximal ideal of $\mathbf{T}_{503}$ corresponding to the 2-torsion in any of these curves. An explicit computation of $S_2(\Gamma_0(503))$ and the eigenvalues mod 2 of the Hecke operator T_{11} shows that there can be at most one other eigenform f_4 of level 503 giving rise to ρ, and indeed one can check that such a form f_4 exists, defined over $\mathbf{Q}_2$ but not $\mathbf{Q}$ (one need only check congruence for the first 84 coefficients). The completion of $(\mathbf{T}_{503})_{\mathfrak{m}}$ now can be checked to be isomorphic to the subring of $(\mathbf{Z}_2)^4$ generated as a $\mathbf{Z}_2$-module by the elements

$$\{(1,1,1,1),(0,2,0,0),(0,0,2,2),(0,0,0,4)\} \subset (\mathbf{Z}_2)^4.$$

The unique maximal ideal of this ring is generated as a $\mathbf{Z}_2$-module by the elements

$$\{(2,0,0,0),(0,2,0,0),(0,0,2,2),(0,0,0,4)\}.$$

We see easily that $(\mathbf{T}_{503})_{\mathfrak{m}}$ is Cohen-Macaulay, as it has Krull dimension 1 and no nilpotent elements. The ideal $\mathfrak{i}$ generated by $(2,2,2,2)$ is a parameter ideal, as $\mathfrak{m}^2 \subseteq \mathfrak{i}$. Finally, the ideals

$$\mathfrak{i}_1 := ((0,2,2,2),(2,0,0,0)) \text{ and } \mathfrak{i}_2 := ((2,0,2,2),(0,2,0,0))$$

have $\mathfrak{i}$ as their intersection, hence $\mathfrak{i}$ is reducible, and therefore the localisation $(\mathbf{T}_{503})_{\mathfrak{m}}$ is not Gorenstein and hence $\mathbf{T}_{503}$ is not Gorenstein. Hence as before, mod 2 multiplicity one fails for $\mathfrak{m}$.

5.4 Other examples

After these initial examples were discovered, William Stein suggested another way to check directly that multiplicity one fails in these and other cases, by an explicit computation in the Jacobian of the relevant modular curve.

Let q be a prime and let E_1 and E_2 denote two non-isogenous new optimal modular elliptic curves (also called strong Weil curves) of conductor q, viewed as abelian subvarieties of the abelian variety $J_0(q)$. Suppose that the two modular forms corresponding to E_1 and E_2 are congruent modulo 2. Let $\mathfrak{m}$ be the corresponding maximal ideal of $\mathbf{T}_q$ over 2. If mod 2 multiplicity one holds for $\mathfrak{m}$, then $J_0(q)[\mathfrak{m}] = E_1[2] = E_2[2]$, viewed as subsets of $J_0(q)$. In particular, $E_i(\mathbf{C}) \bigcap E_j(\mathbf{C})$ is non-zero. But this intersection can be explicitly computed using the `IntersectionGroup` command in Magma, and if it is zero then we have verified that mod 2 multiplicity one has failed without having to compute the completions of the relevant Hecke algebra explicitly.

In the case $q = 431$, we find that $E_1(\mathbf{C}) \bigcap E_2(\mathbf{C}) = \{0\}$, so we have another proof that mod 2 multiplicity one and the Gorenstein property fail for the Hecke algebra of level 431. A similar argument verifies failure of mod 2 multiplicity one at level 503.

This method can also be used to check our third example. There are five isogeny classes of elliptic curves with conductor 2089, of which four, the ones labelled 2089A, 2089C, 2089D, 2089E in Cremona's tables, have isomorphic mod 2 representations. (Note that 2089B, has rational 2-torsion, so its associated Galois representation is reducible.) Using the results of the previous section, we find that the intersection in $J_0(2089)$ of the elliptic curves labeled 2089A and 2089B is $\{0\}$. (Incidentally, the intersection of the curves labeled 2089A and 2089E is $(\mathbf{Z}/2\mathbf{Z})^2$.) Hence we have another example of failure of mod 2 multiplicity one and Gorenstein-ness.

Theorem 5.1 answers the question raised in [5] and in [20], in that it exhibits specific examples of non-Gorenstein Hecke algebras. This raises the natural question of deciding exactly which Hecke algebras are not Gorenstein, without explicitly computing them, and proving theorems like those of Buzzard, Edixhoven, Gross and Mazur for these algebras. A natural next step is to ask the following:

Question 5.4 *Are there infinitely many prime integers q such that $\mathbf{T}_q$ is not Gorenstein, and* *hence where mod 2 multiplicity one fails?*

5.5 Shimura curves and mod 2 multiplicity 1

In forthcoming work of Helm [21], there is a useful observation on the connection between normal modular curves and Shimura curves. He considers the Jacobians of both of these curves, and notes that the Gorenstein property for these curves is connected.

Notation 5.5 *Let N be squarefree, let $\mathbf{T}$ be the Hecke algebra for weight 2 and level $\Gamma_o(N)$ classical cuspidal modular forms, and let $\mathfrak{m}$ be a maximal ideal of $\mathbf{T}$ attached to a newform of level $\Gamma_0(N)$, with residue characteristic ℓ.*
We let D be a positive integer divisor of N, with an even number of prime factors, and we let S be the set of prime divisors of D.

Helm considers the localisation of $\mathbf{T}^{S-new}$ at $\mathfrak{m}$, and under the assumption that it is Gorenstein, and a more technical assumption on the primes dividing N, he proves that the Shimura curve $X_0^D(N/D)$ satisfies mod 2 multiplicity 1 (that is, $J_0^D(N/D)[\mathfrak{m}]$ has dimension 2, as in the definition of multiplicity 1 for modular curves).
This Shimura curve is the one associated to the quaternion algebra of discriminant D, with level structure $\Gamma_0(N/D)$.
He notes that if $J_0^D(N/D)[\mathfrak{m}]$ has dimension 2, then $\mathbf{T}_{\mathfrak{m}}^{S-new}$ is Gorenstein. We call this statement $\blacklozenge$.
Now, in [28], Ribet shows that mod 2 multiplicity 1 does not always hold for Shimura curves, and that the curve $X_0^{2123}(1)$ in particular does not satisfy mod 3 multiplicity 1.
We see in [29], section 3.3.7.1, that there are two elliptic curves of conductor 2071, which is $19 \cdot 109$, and that there is a maximal ideal $\mathfrak{m}$ attached to the modular forms associated to these curves such that $\mathbf{T}_{\mathfrak{m}}$ is not Gorenstein.
By the contrapositive of the statement $\blacklozenge$, we see that the mod 2 multiplicity one condition also fails for $X_0^{2071}(1)$:

Theorem 5.6 *There exists a maximal ideal $\mathfrak{m}$ attached to an elliptic curve defined over $\mathbf{Q}$ of conductor 2071 (the elliptic curve 2071A in Cremona's tables) such that the Hecke algebra $\mathbf{T}_{\mathfrak{m}}$ is not Gorenstein and the Jacobian of the Shimura curve $X_0^{2071}(1)$ does not satisfy mod 2 multiplicity 1.*

This exhibits a new non-multiplicity 1 representation in the Jacobian of a Shimura curve, as Ribet works under the hypothesis that the residue characteristic ℓ is prime to $2N$.
We see that this type of correspondance allows us to move between the modular and Shimura curves, and that this means that the lack of a q-expansion principle can be "worked around".

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